Student Difficulties with Linearity and Linear Functions and Teachers' Understanding of Student Difficulties

by

Valentina Postelnicu

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Graduate Supervisory Committee:

Carole Greenes, Chair
Victor Pambuccian
Finbarr Sloane

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ABSTRACT

The focus of the study was to identify secondary school students' difficulties with aspects of linearity and linear functions, and to assess their teachers' understanding of the nature of the difficulties experienced by their students. A cross-sectional study with 1561 Grades 8-10 students enrolled in mathematics courses from Pre-Algebra to Algebra II, and their 26 mathematics teachers was employed. All participants completed the Mini-Diagnostic Test (MDT) on aspects of linearity and linear functions, ranked the MDT problems by perceived difficulty, and commented on the nature of the difficulties. Interviews were conducted with 40 students and 20 teachers. A cluster analysis revealed the existence of two groups of students, Group 0 enrolled in courses below or at their grade level, and Group 1 enrolled in courses above their grade level. A factor analysis confirmed the importance of slope and the Cartesian connection for student understanding of linearity and linear functions. There was little variation in student performance on the MDT across grades. Student performance on the MDT increased with more advanced courses, mainly due to Group 1 student performance. The most difficult problems were those requiring identification of slope from the graph of a line. That difficulty persisted across grades, mathematics courses, and performance groups (Group 0, and 1). A comparison of student ranking of MDT problems by difficulty and their performance on the MDT, showed that students correctly identified the problems with the highest MDT mean scores as being least difficult for them. Only Group 1 students could identify some of the problems with lower MDT mean scores as being difficult.

Teachers did not identify MDT problems that posed the greatest difficulty for their students. Student interviews confirmed difficulties with slope and the Cartesian connection. Teachers' descriptions of problem difficulty identified factors such as lack of familiarity with problem content or context, problem format and length. Teachers did not identify student difficulties with slope in a geometric context.

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Chapter 1

OVERVIEW OF THE STUDY

1.1 Introduction

Algebra I has been referred to as the "gateway to a technological future" because of its place as the first course in the pipeline to the study of more advanced mathematics, science, and technology (Katz, 2007a). Since the early 1990s and the beginning of the *Algebra for All* movement, student difficulties with algebra and algebraic thinking have been studied and well documented. Those difficulties have persisted despite an increase in remediation efforts nationwide (Caglayan & Olive, 2010; Greenes & Rubenstein, 2008; Kaput, Carraher, & Blanton, 2008; Kieran & Chalouh, 1993; Kilpatrick & Izsák, 2008; Moses B., 1999; Rowntree, 2009). Failure rates in Algebra I have ranged from 16% to 54%, with higher percentages associated with socio-economically disadvantaged students (Bracey, 2000; Dietz, 2010; Gamoran & Hannigan, 2000; Helfand, 2006 ; Johnson, 2008).

In efforts to gain insight into the nature of difficulties with algebra, some researchers have studied students' understanding of the fundamental concepts, linearity and linear functions (Ben-Chaim, James, Fitzgerald, & Benedetto, 1998; Lanier, 1999; Lobato & Siebert, 2002; Moschkovich, 1992, 1999; Moschkovich, Schoenfeld, & Arcavi, 1993; Schoenfeld, Smith III, & Arcavi, 1993; Tourniaire & Pulos, 1985; Zaslavsky, Sela, & Leron, 2002). Other researchers have studied teachers' understanding of students' algebraic thinking

(Nathan & Koedinger, 2000) particularly the source and nature of the difficulties (Ding, 2007; Thompson A. & Thompson P., 1996; Thompson P. & Thompson A., 1994). There are indications that teachers are not aware of the types of difficulties students have with basic concepts of algebra, and their predictions of difficult problems for students do not match students' perceptions of difficulty or their performance (Greenes et al., 2007b).

Statement of the Problem. The dual purpose of this study was to identify the difficulties students in Grades 8, 9, and 10 experience while solving problems that require understanding of aspects of linearity and linear functions, and to assess mathematics teachers' understanding of the nature of the difficulties experienced by their students.

Research Questions. This study addressed the following questions:

1. Does student performance on the Mini-Diagnostic Test (MDT) vary by:
    a. Grade level?
    b. Current mathematics course?
    c. Problem?
2. Does student ranking of problems by difficulty, after completing the MDT, correspond to their performance on the MDT?
3. Does teacher ranking of problems by difficulty, after completing the MDT, correspond to student performance on the MDT?
4. What is the nature of students' difficulties?

5. Do teachers' speculations about the nature of students' difficulties correspond to actual student difficulties as identified from student performance on the MDT and interviews?

1.2 Definition of Terms

Linearity. To a mathematician, linearity refers to the property of a function to be homogenous and additive, i.e., $f(mx) = mf(x)$, $f(x_1 + x_2) = f(x_1) + f(x_2)$. The function $f : \Re \to \Re, f(x) = mx, m \neq 0$ (or $y = mx$) fulfills both conditions and can be seen as a relationship of direct variation with a constant rate of change *m*, or a relationship of proportionality with the constant of proportionality *m*. A proportion $\frac{a}{b} = \frac{c}{d}$ can be seen as a relationship of direct variation, or linearity between two variables (Freudenthal, 1983). Linearity appears as direct variation or direct proportionality in Algebra I textbooks used by the participants in both the pilot study and this full study (Burger et al., 2007; Collins et al., 1998; Larson, Boswell, Kanold, & Stiff, 2004). Larson et al. (2004) describe a model of direct variation: "When two quantities *y* and *x* have a constant ratio *k*, ($k \neq 0$), they are said to have direct variation. The constant *k* is called constant of variation. If $\frac{y}{x} = k$, then $y = kx$ " (p. 236).

Linear Function. A linear function is represented in Algebra I programs in the form $y = mx + b$, and is referred to as a linear equation of two variables, the slope-intercept form of the equation of a line, or simply the equation of a line.

Larson et al. (2004) define the linear equation in *x* and *y* as the equation that can be written as "*Ax* + *By* = *C* where A and B are not both zero" (p. 209). The same definition is used Burger et al. (2007, p. 296) and by Collins et al. (1998, p. 281).

Graph of a Linear Function. The graph of a linear function $f(x) = mx + b$ or $y = mx + b$ (the slope-intercept-form) is a line. When $b = 0$, the line contains the origin of the Cartesian system of coordinates. When $b \neq 0$, the line crosses the *Y*-axis at $(0, b)$, a point different from the origin.

***X*-intercept, *Y*-intercept.** The *x*-intercept (or the "zero") of the function $f(x) = mx + b$ is found by solving the equation $f(x) = 0$; the solution is $x = -\frac{b}{m}, m \neq 0$. In a Cartesian system of coordinates, the point $(-\frac{b}{m}, 0)$ represents the point of intersection between the graph of a function G_f and the *X*-axis. The *y*-intercept of the function $f(x) = mx + b$ is found by calculating the value of the function $f(0)$; the solution is $f(0) = b$. The point $(0, b)$ represents the point of intersection. In Algebra I textbooks, definitions for the *x*-intercept and *y*-intercept refer to their graphic representation; the *x*-coordinate of a point where a graph crosses the *X*-axis and the *y*-coordinate of a point where a graph crosses the *Y*-axis (Larson et al., 2004, p. 222). The same definitions are found in Burger et al. (2007, p. 296) and Collins et al. (1998, p. 346).

Slope or Rate of Change. The slope of a line is defined in Algebra I textbooks as "rise over run" and a formula is provided for calculating slope when the Cartesian coordinates of two points, (x_1, y_1) and (x_2, y_2) are given; the

slope is $m = \frac{y_2 - y_1}{x_2 - x_1}$ (Larson et al., 2004, pp. 229-230). Some textbooks use "ratio," "change," and "constant rate of change," in their definitions. For example, Burger et al. (2007) define slope as "the constant rate of change shown by the line," (p. 296) and Collins et al. (1998) define slope as "the ratio of the change in the *y*-coordinate to the corresponding change in the *x*-coordinate" (p. 326). Stump (1997) identifies five different contextual definitions of slope: geometric, algebraic, physical, trigonometric, and functional. Geometrically, slope is defined as a property of a line. Given two points (x_1, y_1) and (x_2, y_2), the slope of the line connecting the two points is $m = \frac{y_2 - y_1}{x_2 - x_1}$. Slope is defined algebraically as one of the parameters in the linear equation; slope is the $\underline{m}$ in the equation $y = \underline{m}x + b$. Slope describes the steepness of physical objects such as mountain roads, ski slopes, and wheelchair ramps. Slope is defined trigonometrically as the tangent of the angle of inclination θ; slope is $m = \tan\theta$. When linear functions, of the form $y = mx + b$, are understood to be composed of two variables, *x* and *y*, that have a dynamic relationship, slope represents the constant rate of change $m = \frac{\Delta y}{\Delta x}$ (Stump, 1997, p. 4).

Cartesian Connection. The Cartesian connection can be stated mathematically with the following theorem: A point (x_0, y_0) is on the graph of the line *l* if and only if its coordinates satisfy the equation of *l*, $y = mx + b$

(Knuth, 2000; Moschkovich et al., 1993; Schoenfeld et al., 1993). Schoenfeld et al. (1993) consider all mathematical conventions associated with graphic representations in a Cartesian system of coordinates to be part of the Cartesian connection, as for example, the association between the change in *y* and the change in *x* from the slope formula, and their graphic representations as line segments with oriented magnitudes, rise and run.

1.3 Rationale

1.3.1 The Need for More Scientists. To compete successfully internationally with scientific and technological advances, the United States must increase its innovation capacity and produce more experts in mathematics, technology, engineering, physics and the natural sciences (Couto, Mani, Lewin, & Peeters, 2007; Lewin & Couto, 2006; National Science Board, 2008, 2010). Since the late 1990s, the United States has lost its leadership capacity in these areas. Data from the National Science Board (2008) and the Business Higher Education Forum (2005) show an increasing number of advanced degrees in science and engineering being awarded in China, India and Europe. The United States "has lost ground in many areas: college degrees earned in STEM fields; scientific journal articles authored; patents awarded; and Nobel Prizes won" (Business-Higher Education Forum, 2005, p. 3).

The national need for mathematicians and scientists places great responsibility on elementary through high school and college programs to engage more students in the study of these subjects (Committee for Economic Development, 2003;

Public Works, 2006). National and state organizations interested in increasing the number of workers with 21st century skills have recommended that students take increasingly challenging mathematics courses throughout their school programs, and complete, at minimum, Algebra I, Geometry, and Algebra II before high school graduation (ACT, 2006; American Diploma Project, 2004; Arizona Community Foundation, 2008; Bozick & Lauff, 2008; Darling-Hammond, 2007; Education Week, 2006; National Center for Public Policy and Higher Education, 2004; Partnership for 21st Century Skills, 2007). Research studies examining the relationship of high school course-taking histories to graduation from high school, entrance and graduation from college, and successful entry into the workforce, found that two courses, Algebra I and Geometry, had the highest correlation (Pelavin & Kane, 1990; Shettle et al., 2007).

1.3.2 Algebra, the Primary Course in the Pipeline to STEM Careers.

Several professional associations, including the National Council of Supervisors of Mathematics (NCSM) (1977), the National Council of Teachers of Mathematics(NCTM) (1980; 1989; 2000) and the Mathematical Association of America (Katz, 2007a), have recognized the value of algebra to the study of higher level mathematics, physics, the natural sciences, engineering, and technology (Katz, 2007a; Ma, 2009; Sadler & Tai, 2007). In its *Position Paper on Basic Mathematical Skills,* the NCSM (1977) identifies algebra as a pivotal discipline in the school mathematics curriculum. In its *Principles and Standards for School Mathematics,* the NCTM (2000) identifies algebra as one of five

major strands in mathematics for study by students Prekindergarten through Grade 12, and highlights the importance of knowledge of algebra and abilities to think algebraically to the successful study of other mathematical domains including geometry, probabilities and statistics, calculus.

In 1992, the National Assessment of Educational Progress (NAEP) reported that Grade 8 students enrolled in Algebra I had higher scores in mathematics on the assessment than did Grade 8 students enrolled in Pre-Algebra, and Pre-Algebra Grade 8 students had higher NAEP scores than did students enrolled in general mathematics courses (National Center for Education Statistics, 1996). Given the NAEP results plus recognition of the importance of algebra in the pipeline preparation for STEM careers, educators and others began to promote algebra as a course for all students. In February 1994, the *Algebra for All* movement was born (Pennsylvania Council of Teachers of Mathematics, 1997). Throughout the United States, all students in Grades 8 or 9 were required to enroll in Algebra I. From 1996 to 2005, the proportion of U.S. students taking Algebra I in Grade 8 rose from 24% to 34% (Cavanagh, 2008). By 2007, nationwide 31% of Grade 8 students were enrolled in Algebra I (Loveless, 2009). These percentages are expected to rise as more states, following Minnesota's and California's examples, requiring all Grade 8 students to enroll in Algebra I (Jacobson, 2008; Loveless, 2009).

1.3.3 Addressing Failure with Algebra I. Shortly after the mandate of *Algebra for All*, publications began appearing that identified huge failure rates in Algebra I courses across the country (Gamoran & Hannigan, 2000; Kilpatrick & Izsák, 2008; Dietz, 2010).

Even the most successful school districts participating in the College Board Equity 2000 program, reported failing rates of 25% (Gamoran & Hannigan, 2000). Roy Romer, Los Angeles Unified School District superintendent between 2001 and 2006, commented in January 2006 that Algebra I "triggers (school) dropouts more than any single subject" (Helfand, 2006). In 2004 in Los Angeles schools, the failure rate in Algebra I was 44% (Helfand, 2006). In 2005-2006, the Algebra I failure rate overall in Virginia was 19.1%. The rate was 10% higher for economically disadvantaged (26.6%) versus economically non-disadvantaged students (16.7%) (Johnson, 2008). In 2009, 55% of New Jersey students (Carroll, 2009), and 84% of Clark County, California students failed end-of-semester Algebra I tests (Richmond, 2009).

Algebra I is foundational to the study of subsequent high school and college mathematics courses. For this reason, the large failure rate prompted much research about student difficulties with key Algebra I concepts. Among these concepts are linearity and linear functions, which have been identified by the National Mathematics Advisory Panel as major topics of school algebra (National Mathematics Advisory Panel, 2008). As noted by Kalchman and Koedinger (2005), understanding the x-intercept, y-intercept and how to graph linear functions, offers to students the possibility of comparison, differentiation,

and generalization to other functions like quadratic, exponential, and trigonometric functions, and enhances their learning of those non-linear functions. Not only are these concepts important for preparation for higher level mathematics, they also are central to the solution of problems involving measurement conversion and similarity in the pre-secondary school curriculum, linear models and approximations in calculus and statistics, and linear programming and vectorial spaces in secondary and post-secondary courses (De Bock, Van Dooren, Janssens, & Verschaffel, 2002). As noted by Thompson (1994a, 1994b), a mature conceptual understanding of rate of change as "reflective abstract ratio" has the potential to be foundational for the understanding of calculus concepts.

Many educators believe that student difficulties with Algebra stem from the instructional methods and curriculum itself. To better prepare Algebra students the curriculum "must simultaneously develop conceptual understanding, computational fluency, and problem-solving skills" recommends the National Mathematics Advisory Panel (2008, p. xix). Despite long-term efforts to consider alternative delivery methods for instruction (Rettig & Canady, 1998) and research to gain greater insight into conceptual difficulties (Chazan, 2008; Doerr, 2004; Greenes & Rubenstein, 2008; Kaput, Blanton, & Moreno, 2008; Kieran, 1992; Moses, 1999) failure rates in Algebra I persisted (Kilpatrick & Izsák, 2008).

1.3.4 Studies of Student Difficulties with Linearity and Linear Functions in Grades 8-10. Several researchers studied student difficulties with key concepts of Algebra I, particularly slope or rate of change (Lobato & Siebert, 2002), the *x*- and *y*-intercepts (Moschkovich, 1992, 1999; Moschkovich et al., 1993; Schoenfeld et al., 1993; Smith, Arcavi, & Schoenfeld, 1989), and linear relationships (Ellis, 2007, 2009).

Lobato and Siebert (2002) examined student understanding of slope in physical situations as a measure of steepness of a ramp, and slope in functional situations as a measure of speed or rate of change. Nine students from Grades 8-10 with average mathematical performance participated in this study. Analyses revealed student difficulties in making connections between the physical aspect of slope and the functional aspect as rate of change. Lobato and Siebert concluded that students may conceive of the speed of a mobile differently from the steepness of a ramp, despite the fact that, mathematically, speed magnitude and slope as a measure of steepness may be expressed by the same formula. Lobato and Siebert's study pointed to the importance of context in the conceptualization of slope, and student difficulties transferring knowledge between contexts, often overlooked by curriculum developers and instructors.

Researchers from the Functions Group at Berkeley used a graphing software tool specifically designed as an environment for the learning of functions, to study Grades 8-10 students' abilities to make connections among various representations of linear functions. Participants were observed and interviewed while interacting with the software during various tasks (Schoenfeld et al.,

1993), and observed during their classroom work and participation in peer discussions (Moschkovich 1992, 1999; Moschkovich et al., 1993). Findings highlighted the crucial importance of the Cartesian connection (Moschkovich et al., 1993) and revealed student difficulties with making connections among representations. The studies mentioned above (Lobato & Siebert, 2002; Moschkovich, 1992, 1999; Moschkovich et al., 1993; Schoenfeld et al., 1993; Smith et al., 1989) were conducted by researchers in university laboratories. It also should be noted that classroom teachers, indispensible to the natural teaching-learning environments, did not participate in any of these studies.

Davis (2007) analyzed eight sophomore and junior students' understanding of the *y*-intercept in real-world contexts and across multiple representations over a period of 10 weeks of instruction while they were enrolled in Algebra II. During this time, and with an approach that delayed the introduction of formal mathematical terminology, and allowed students to use their own terminology, Davis observed and interviewed the students. Analyses of student performance revealed a weak connection between various representations of the *y*-intercept and the use of student-invented terminology. Davis explained these findings as being influenced by students' prior knowledge of the *y*-intercept in real contexts, and the teacher's conception of the *y*-intercept and his assumptions that students would have no difficulty understanding the relationship between the formal mathematical language and the informal language suggested by the real-world contexts. The use of informal phrase "starting point" for the *y*-intercept suggested by the real-world contexts, was not be translated to symbolic, tabular,

and graphic representations, and seemed to have a negative impact on student performance on abstract tasks, i.e., tasks without real-world contexts. Although the sample was small, the study showed that student difficulties with connecting among various representations persisted even with instruction. The relationship between student difficulties and teacher understanding of these difficulties was analyzed by Davis who pointed to the teacher's insensitivity to students' difficulties. Davis concluded that the teacher's unquestioned assumptions about student abilities to make connections between representations and transfer knowledge of the *y*-intercept between contexts, as well as lack of appropriate instructional materials, impacted negatively on student learning.

In summary, studies on our target population (Grades 8-10) suggest that linearity and linear functions are complex topics, and student difficulties occur most often with making connections among various representations of linear relationships and functions (Davis, 2007; Moschkovich, 1992, 1999; Moschkovich et al., 1993; Schoenfeld et al., 1993; Smith et al., 1989). Furthermore, misconceptions are robust (Moschkovich, 1992, 1999; Moschkovich et al., 1993; Schoenfeld et al., 1993; Smith et al., 1989). With the exception of one study referring to teacher understanding of student difficulties with the *y*-intercept (Davis, 2007), all other studies addressed only student difficulties with aspects of linearity and linear functions, as identified by researchers. None of the studies asked students to explain the nature of their difficulties, or compared student difficulties with teachers' insights into their students' difficulties.

1.3.5 Studies with other Populations. Several studies, conducted with other populations, including elementary and middle school students, upper level high school students, college students, and pre-service teachers, focused on difficulties with the concepts of slope and rate of change, connections among various representations of functions, and the modeling of phenomena or real world situations using functions. Research on proportional reasoning, considered to be equivalent to linear reasoning (De Bock, Verschaffel, & Janssens, 1998; De Bock et al., 2002; De Bock, Van Dooren, Janssens, & Verschaffel, 2007; Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003; Van Dooren, De Bock, Janssens, & Verschaffel, 2007), found that students have great difficulty with reasoning proportionally, and difficulties persist throughout high school and college (Ben-Chaim et al., 1998; Post, Behr, & Lesh, 1988).

Li (2006) studied a group of 317 students in Grades 6 and 7 from two states in order to identify difficulties and errors exhibited as the students completed tasks focusing on the concepts of variable, linear equation, and linear function. After analyzing students' systematic errors, faulty algorithms, and misconceptions, a comparison between high-achieving students and low-achieving students pointed to the possible causes of students' misconceptions. The high-achieving students had an object or structural view of the mathematical concepts while the low-achieving students had a process view. For example, when asked to solve problems like, "Look at the equation $y = 2x + 5$; if x increases by 1, what happens to y?" (Li, 2006, p. 169), students with a process view of the linear function would use a table of values (input-

output table) to answer the question, while students with an object view of the linear function would make a prompt judgment by looking at slope. Li suggested that the persistence of difficulties might be caused by the use of the recycling curriculum in which the same topics are presented in the same way over several years.

At the Precalculus level, two studies are of particular relevance. Stump (2001a) interviewed 22 Precalculus students to gain insight into their understanding of slope as a measure of steepness in physical situations, and as a measure of rate of change in functional situations. Findings confirmed that student difficulties with the concept of slope are robust. In physical situations, students used angles instead of ratios as a measure of steepness. They performed better in functional situations, but could not connect the rate of change of a function to the slope of the line representing that function. Students' use of informal language in describing steepness (e.g., "angle", "incline", "height"), and describing the rate of change (e.g., "The more tickets you sell the more profit you make") was associated with less conceptualization of slope as ratio or rate.

Hauger (1997) conducted interviews of four high school students enrolled in a Precalculus course to verify student difficulties with the concept of rate of change previously reported in the literature, and to confirm the importance of the linguistic and graphic connection when modeling change with functions. Hauger's students had difficulty making connections between linguistic (e.g., "steady pace") and graphic representations of functions. Often, the language

used by teachers in instruction or the textbooks themselves contains words and expressions loaded with meanings considered so transparent that they do not need explanation. Yet these terms or phrases remain opaque to students. For example, the constant rate of change of a linear function represented graphically by a static line might suggest to students that no change occurs.

Zaslavsky et al. (2002) hypothesized that the connection between the symbolic (algebraic/ analytic) and the graphic (geometric/ visual) aspects of slope becomes problematic when non-homogenous systems of coordinates are used. The 124 participants in their study included Grade 11 students, prospective and in-service secondary mathematics teachers, and university mathematics educators and mathematicians. Participants were interviewed about two tasks requiring them to analyze slope appearing as an analytic attribute of a linear function, and as a geometric attribute of the line representing the graph of the function. Task 1 required participants to identify the slope of a line graphed in a homogenous system of coordinates. Task 2 required participants to identify the slope of a line graphed in a non-homogeneous system of coordinates. Results revealed that participants with strong mathematical backgrounds could reconcile the analytic and the geometric perspectives, while others experienced difficulties in connecting the two perspectives or favored only one of them. This study draws attention to the unquestioned assumptions behind the graphic representations, and the need for teachers and instructional materials to make explicit the origin, sense, and the unit of measurement for a Cartesian system of coordinates. It also points out that higher levels of mathematical knowledge are

associated with higher rates of success with connecting geometric and analytical perspectives of slope.

A longitudinal study of several years duration, focusing on participants' understanding of the use of graphs to represent physical concepts or real–world situations, was conducted with college students, high school students and pre-service teachers enrolled in physics courses (McDermott, Rosenquist, & Van Zee, 1987). The study revealed that the nature of the participants' difficulties with graphing were common across the various populations studied. In particular, participants had difficulty connecting graphs to physical concepts and to real world situations.

The National Survey of Algebra Teachers (Hoffer T. B., Venkataraman, Hedberg, & Shagle, 2007) points out that solving word problems is considered by Algebra teachers to be their students' poorest area of preparation, but does not reveal more on the nature of student difficulties. As indicated by McDermott et al.'s (1987) study, the source of the trouble may be the difficulty of connecting between situations, concepts, and their representations.

Studies by Stump (1997, 2001a, 2001b), Hauger (1997), Zaslavsky et al. (2002), McDermott et al. (1987) described the difficulties experienced with linear functions by various populations (high school students, pre-service teachers, and in-service teachers). The prevalence of the same difficulties in these populations as well as the potential deleterious effect on the study of physics, further reinforces the need for the proposed study. Moreover, difficulties with slope and connecting among representations seem to be

common to both students and teachers, and raise the question of teachers' understanding of these concepts as well as their students' difficulties with these concepts.

1.3.6 Studies of Teacher Understanding. Research on student and teacher understanding of proportionality (Tourniaire & Pulos, 1985) and the concept of function, in general, is rich and reveals students' and teachers' difficulties (Breidenbach, Dubinky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Hansson, 2006; Norman, 1992; Sfard, 1992; Sierpinska, 1992; Slavit, 1997; Vinner, 1992). However, the research on teacher understanding of student difficulties, specifically with linearity and linear functions, is scarce. Several studies confirm the strong correlation between teachers' subject matter knowledge and pedagogical knowledge (Stump, 1997; You, 2006) and the importance of teacher knowledge on addressing student difficulties successfully (Ding, 2007).

Using interviews and surveys, Stump (1997) examined the mathematical and pedagogical knowledge of slope of 39 pre-service and in-service secondary mathematics teachers. Results showed that most of the teachers had a geometric view of the concept of slope. The algebraic, physical, functional, trigonometric or ratio aspects of slope were less mentioned in their descriptions of the concept. Teachers' pedagogical knowledge mirrored their subject matter knowledge. They often mentioned student difficulties related to the geometric representation of slope, like difficulties with the graphing of lines. What was not considered by Stump was how the nature of teachers' knowledge of slope correlates with their

students' knowledge of the concept, nor whether teachers' speculations about their student difficulties with the concept of slope correspond with actual student difficulties.

Learning cannot be separated from teaching, and many of the studies mentioned in this chapter focused solely on students' understandings and difficulties with linearity and linear functions, irrespective of teachers' understanding of the nature or extent of these difficulties. The present study was conducted to reveal specific patterns in students' understandings and difficulties with linearity and linear functions and to examine teachers' understanding of their students' difficulties. The results of this study have the potential of better informing researchers and mathematics educators in the design of instructional materials as well as interventions for addressing the problematic aspects of students' difficulties with linearity and linear functions.

Moreover, results of this study have the potential of contributing to teachers' understanding of students' difficulties with fundamental algebraic ideas in order to enable them to better respond to the mathematical talents and needs of their students.

Chapter Summary. This chapter presented an overview of the study of student difficulties with linearity and linear functions and teachers' understanding of student difficulties, followed by the statement of the problem and the research questions (Section 1.1), definition of terms used to describe linearity and linear functions (Section 1.2), and rationale for this study (Section

1.3). In the next chapter, research literature to support this study is discussed and analyzed.

Chapter 2

REVIEW OF LITERATURE

2.1 Introduction

The purpose of this study was to investigate student difficulties with concepts of linearity and linear functions, and compare those to teachers' assessments of their students' difficulties.

In this chapter, an historical perspective of algebra as a body of knowledge (Section 2.2) and algebra as a course or strand in the school curriculum are presented first (Section 2.3). This is followed by a review of the research literature on student difficulties and teacher understanding of rate of change, the fundamental concept of linearity and linear functions, together with research studies on student difficulties with the Cartesian connection (Sections 2.4 and 2.5). The chapter continues with studies on student and teacher assessment of problem difficulty (Section 2.6), and concludes with several remarks (Section 2.7).

2.2 Algebra as a Body of Knowledge: Historical Perspective

In 1770, Euler defined algebra by stating its purpose, as "the science which teaches how to determine unknown quantities by means of those that are known" (Euler, 1984, p. 270). Cooke (1997) elaborated on Euler's definition to consider parts of algebra to be problems that require finding unknown quantities from given quantities without indicating what operations should be performed on the given quantities, even in the absence of symbolic representations.

These definitions of algebra can be traced back more than four millennia ago to the times of the ancient Akkadian Empire (2300 BC – 2200 BC). At that time, problems involving linear and quadratic relationships were solved not by formulas, but by examples. Babylonian clay tablets, dated from the twentieth century B.C. to the seventieth century B.C., described solutions to problems requiring the length and the width of a rectangle if certain relationships were given (e.g., the sum and the product of the length and the width are known) (Katz, 2007b). This stage in the development of algebra, known as the non-symbolic stage (Katz, 1998; Cooke, 1997), was characterized by the use of natural language (e.g., words, sentences) and some abbreviations to express the rules used for finding quantities to satisfy certain conditions.

The development of algebra as a body of knowledge was marked throughout the centuries by the quest for solving equations. During the third century, Diophantus (c. 200 - c. 284) gave an algorithm for solving quadratic equations. In the ninth century, Al-Khwarizmi (c. 780 - c. 850) wrote a manual for solving abstract linear and quadratic equations based on the Babylonian algorithms used for solving real-life problems. In the eleventh century, Omar Khayyam (1048-1131) discovered a general method for solving cubic equations by employing the Greek idea of intersecting conics. In the sixteenth century, Cardano (1501-1576) published *Ars Magna* (1545) which contained algorithms for finding solutions to third and fourth degree polynomial equations that were based on geometric proofs. Throughout this non-symbolic stage of algebra, since the Akkadian

Empire until the European Renaissance, the justifications for the algorithms proposed for solving equations appeared to be mainly geometric in nature.

The rise of algebra during the European Renaissance, between the fourteenth and seventeenth centuries, marked the entrance of the symbolic stage with François Viète's (1540-1603) work on equations, characterized by the use of symbols to represent quantities (e.g., vowels for unknown quantities and consonants for known quantities). Over time all quantities and all operations and relationships involving quantities were symbolized, culminating with the development of rules for symbolic manipulation.

The shift from the static approach for solving equations to the dynamic function approach developed during the seventeenth century with the work of Galileo (1564-1642), Kepler (1571-1630), and Newton (1643-1727) during their attempts to describe the movement of projectiles and the solar system. The focus shifted from searching for numbers that could be solutions of equations to searching for equations of curves to represent paths of moving objects. Justifications for descriptions of the paths of mobiles were difficult to communicate and remained geometric in nature until the work of Fermat and Descartes (1637) in analytical geometry facilitated representation of motion using a curve.

Algebra entered its abstract structural stage during the nineteenth century, and started to depart from its aim of solving equations. The attempts of Lagrange (1736-1813), Abel (1801-1829), and Galois (1811-1832) to solve polynomial equations of degrees higher than four led to the development of group theory.

Abel used Lagrange's idea of permutations to prove that there is no general solution to a polynomial equation of the fifth degree. Galois invented group theory to determine conditions needed to be fulfilled in order for a polynomial equation to have a solution.

2.3 Algebra in the School Curriculum

To understand how school algebra, and in particular the Algebra I curriculum was shaped in the United States, an historical perspective is presented first (Section 2.3.1), followed by a curriculum perspective of school algebra (Section 2.3.2).

2.3.1 Algebra in the School Curriculum: Historical Perspective. The rise of algebra during the European Renaissance marked the beginning of algebra as a course of study at universities, and the creation of algebra textbooks. The first algebra textbook written in vernacular, Luca Pacioli's *Summa de Arithmetica, Geometria, Proportioni et Proportionalità,* was published in Venice in 1494 (Rey, 2006).

In the United States, Algebra as a course of study began during the eighteenth century as a college course that focused on how to solve linear and quadratic equations with numerical coefficients, and how to simplify simple algebraic expressions (Kilpatrick & Izsák, 2008). The Benjamin Franklin Academy (later the University of Pennsylvania) had an algebra course as part of its curriculum in 1751. By 1820, Harvard implemented an algebra entrance examination, and in time other colleges started to do so, as well. After the Civil War, geometry began

to be required for admission, thus setting in stone the algebra-geometry sequence in the secondary curriculum (Kilpatrick & Izsák, 2008).

In 1827, Massachusetts passed America's first high school law that mandated the teaching of algebra, geometry and surveying in all high schools in every town with more than 500 families. The content of the secondary algebra course was initially shaped by practical, vocational purposes. Slowly, the college algebra course content became a secondary school curriculum. The practical (vocational, technical) approach to algebra disappeared by the end of the nineteenth century, being replaced by the academic approach – to meet college entrance requirements. The content of the high school algebra course was dominated by exercises in simplifying algebraic expressions and solving equations of increased complexity (Kilpatrick & Izsák, 2008).

During the first half of the twentieth century, the utility of algebra for the average student was seriously questioned. Consequently, the algebra course content was revised to reflect the ideal of social efficiency (Kliebard, 2004). More attention was given to using algebraic formulas, and interpreting graphs, and less attention was given to skills like solving quadratic equations or symbolically manipulating complex algebraic expressions.

For a short period during the New Math era, from the mid 1950s to the mid 1970s, the algebra curriculum was radically changed. As a consequence of the Sputnik crisis (1957), the new curriculum's aim was to better prepare students for college in order to preserve the United States' place as world leader in space technology. The emphasis on structural properties of the fields of real numbers

and complex numbers, and the use of key unifying ideas like sets, functions and relations, characterized the New Math algebra curriculum. Proofs also became part of the algebra content because of their potential to help students understand the nature and role of deductive reasoning in mathematics.

Of interest is the influence of Dieudonné and Piaget during the New Math era. Dieudonné viewed mathematics as a unified whole, a "totality" kept together by Bourbaki's "mother-structures"– topological structure, order structure, and algebraic structure (Corry, 2004). Dieudonné advocated for a unified structural perspective in the mathematics curriculum and the elimination of the Euclidean geometry, as something that escaped this unification. Piaget's (1971) "natural order" of intellectual development, topological to Euclidian intuitions, sustained this point of view.

The New Math curriculum failed in the United States (Kline, 1973). The end of the New Math era meant the end of the structural approach in the secondary school algebra, and the end of Euclidean geometry as a course of study. The Back-to-Basic movement in the 1970s returned the algebra curriculum to a focus on the "basic" algorithms applied to solving simple equations. The algebra course was extended to two years, and separated in two parts, Algebra I and Algebra II, to make the introductory course, Algebra I, more accessible to students (Kilpatrick & Izsák, 2008).

The *Algebra for All* movement (Pennsylvania Council of Teachers of Mathematics, 1997) advocated for equity and access to algebra for all students. Moses (1995) and Mason (2008) considered that algebra should be treated as a

"new civil right," and "a natural birthright of all human beings," respectively. In 1989, the National Council of Teachers of Mathematics (NCTM) published *Curriculum and Evaluation Standards for School Mathematics*. NCTM revised the earlier document and produced the *Principles and Standards for School Mathematics* in 2000. In both Standards documents, algebra was established as a major strand of the elementary through high school curriculum. Studies of student achievement documented benefits of students' universal and early access to algebra for their future academic achievement (Smith, 1996; Gamoran & Hannigan, 2000; Speilhagen, 2006). In the 1990s, Algebra I became the mandatory "gateway" course to the study of mathematics and higher level mathematics and sciences (Katz, 2007a; Matthews & Farmer, 2008), and started to be pushed from Grade 9, where it has been positioned for a century, to Grade 8 (Kilpatrick, Mesa, & Sloane, 2007).

The shift from the static equation solving to the dynamic functional approach was the most notable conceptual change in the algebra secondary curriculum brought on by the standards-based era. This new algebra curriculum is characterized by an emphasis on the functional approach to algebra, and increased attention to integration and connectedness across mathematical disciplines, ideas, concepts, and representations. Solving equations using a functional perspective coincided with students' broad access to graphing technology (e.g., graphing calculators, computers with graphing software). The algebra curriculum began to include functions and their properties for purposes other than solving equations. Functions started to be used to model change, although on a small scale

(Kilpatrick et al., 2007). The standards-based era culminated with the publication of the *Common Core State Standards for Mathematics* (CCSSO & NGA, 2010) highlighting *modeling* as a fundamental high school mathematics objective.

2.3.2 School Algebra from a Curriculum Perspective. School algebra begins when students use letters to represent numbers. As noted by Usiskin (1988), school algebra may be regarded as the study of generalized arithmetic, algorithms, functions, and structures. Each of these is described below, and is accompanied with MDT problems that are illustrative of each of these four topics.

1. *Generalized arithmetic,* where variables are used as pattern generalizers. For example the pattern 12, 18, 24, 30 can be generalized to $6(n+1)$ or $6n+6$ (see Appendix A, MDT problem 5).
2. The study of *procedures* or *algorithms* for solving certain kinds of problems in which values of variables are often unknowns to be found. For example:
   a) To compute the slope of a line graphed in a Cartesian system of coordinates, two points on the line (x_1, y_1) and (x_2, y_2) must be identified, and the ratio $m = \frac{y_2 - y_1}{x_2 - x_1}$ must be calculated (see Appendix A, MDT problem 4).
   b) To complete a table of values for a given function (e.g., $y = mx + b$), x-values from the domain of the function must be identified, and

corresponding values for *y* must be calculated (see Appendix A, MDT problem 2).

c) To check if a point (x_0, y_0) is situated on the line $y = mx + b$, the coordinates of the point must be substituted into the equation of the line, and a determination must be made whether the statement $y_0 = mx_0 + b$ is true (see Appendix A, MDT problem 1).

3. The study of *relationships among quantities*, where variables are arguments or parameters of *functions*. For example, to recognize the relationship between the rate of change $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ of the linear function $f : \Re \to \Re, f(x) = mx + b, m \neq 0$ (or $y = mx + b$) and the slope $m = \frac{y_2 - y_1}{x_2 - x_1}$ of the line with the equation $y = mx + b$, and make the connection between the symbolic representation and the geometric representation (see Appendix A, MDT problem 7).
4. The study of *structures*, where variables are arbitrary marks on the page or objects that should be manipulated according to the rules of the algebraic symbolic game. For example, apply the rules for simplifying algebraic expressions and obtain equivalent equations in order to rewrite the linear equation from standard form $ax + by = c$ in slope-intercept form $y = mx + b$ (see Appendix A, MDT problem 6).

2.4 Research on Student Understanding of Linearity and Linear Functions

Various research studies have focused on student understanding of aspects of linearity and linear functions. Of interest to this study are student knowledge and difficulties with slope or constant rate of change, and with connections among representations of linear functions, particularly the Cartesian connection. Studies conducted with students in Grades 8-10 and other age groups are presented (Section 2.4.1), followed by teaching experiments (Section 2.4.2), and studies in other content areas (e.g., Physics) (Section 2.4.3).

2.4.1 Studies on Student Understanding of Linearity and Linear Functions.

Greenes, Chang, and Ben-Chaim (2007) conducted a study of 1732 Grades 8 and 9 students enrolled in Algebra I to gain insight into their difficulties with aspects of linearity and linear functions. Grenees et al. (2007b) designed a 7-item assessment, the Mini-diagnostic Test (MDT). The MDT is the same instrument used by Greenes et al. (2007a) and the present study (see Appendix A). The MDT was initially administered to 752 students in Grade 8 in five school districts in the Northeast U.S. Subsequently, the MDT was translated and administered to 405 Grade 8 students in Seoul, Korea, and 575 Grade 9 students in Haifa, Israel. After completion, students were interviewed about their solution processes. Across the three countries, the most difficult problems were MDT problems 4, 7b, and 7c. Moreover, the same difficulties with slope, rate of change or speed persisted across grade levels and instructional programs (i.e., different textbooks). For MDT problem 4, the task is to calculate the slope of a line

graphed in a homogeneous system of coordinates. Only 33.8% of American students, 43.5% of Korean students, and 58.6% of Israeli students successfully solved that problem. MDT problems 7b and 7c require calculation of the slope of a line segment graphed in a non-homogenous system of coordinates (independent variable - time in hours, dependent variable - distance in miles), and the speed of the car, respectively. Of note, MDT problems 7b and 7c refer to the same line segment representing the movement of a car. Percentages of correct answers on MDT problems 7b and 7c were 32% and 48.3%, respectively for the Unites States, 49.9% and 39.5%, respectively for Korea, and 40% and 58.3%, respectively for Israel. Overall, in the three countries, more students correctly answered MDT problem 7c requiring the calculation of speed than MDT problem 7b requiring calculation of slope. American students also had difficulty with MDT problems 1 and 6. For MDT problem 1 students had to decide whether the point $(2,-8)$ is situated on the line $y = 3x - 14$, and explain their reasoning. Only 52.3% of U.S. students provided an explanation. MDT problem 6 requires the identification of the *y*-intercept from five choices (0, 2, 4, 6, and 12), given the equation of the line $2x + 3y = 12$. About one third (35.2%) of U.S. students selected the correct answer for the *y*-intercept.

Interviews of those students revealed difficulties connecting between representations, particularly with the Cartesian connection. In MDT problem 1, students could not connect the point (x_0, y_0) with one of the solutions of the equation $y = mx + b$, or the *y*-intercept with the point $(0, b)$ from the line with the

equation $ax + by = c$. These findings are consistent with those noted by Moschkovich et al. (1993), Smith et al.(1989), and Schoenfeld et al. (1993). In MDT problem 4, students did not connect $x_2 - x_1$ (change in *x* or "run") with its graphic representation in a Cartesian systems of coordinates, an oriented segment with positive sense. This is consistent with findings by Schoenfeld et al. (1993). In addition, in MDT problem 4, students failed to connect the sign of the slope, positive, with the graphed line representing an increasing function. For example, in MDT problem 4, students calculated a negative slope, although the linear function is increasing. This finding is in agreement with research by Moschkovich et al. (1993). In the case of MDT problem 7b, students had difficulties calculating the rise or the change in *y*. The geometric aspect of the slope prevailed for students who answered 4/3 instead of 40, a finding in concert with the work of Zaslavsky et al. (2002). Some students estimated 30 for the *y*-coordinate of the point with the *x*-coordinate 1, and considered the point $(1, 30)$ instead of the lattice point $(3, 120)$ in their calculation of slope.

The present study was inspired by the one conducted by Greenes et al. (2007a). Greenes et al.'s (2007a) study used only descriptive statistics, and neither reported on the internal consistency of the MDT, nor analyzed the dimensions across which the MDT problems aligned. Issues related to the scoring and coding inter-rater agreement were not addressed. A new direction of research was suggested by the fact that the committee of mathematics leaders who designed the MDT speculated on the reasons for student difficulties with aspects of linearity and linear functions, and failed to correctly identify specific difficulties 40% of

the time (Grenees et al., 2007b). This failure raised the question of whether students and teachers could identify the difficulties. Consequently, in the present study all students and their teachers were asked to rank the MDT problems by difficulty, and comment on the nature of student difficulties with the MDT problems.

Knuth (2000) studied high school students' understanding of the Cartesian connection, that is, the relationship between symbolic and graphic representations of linear functions. Participants were 178 students from the same high school, with a traditional curriculum that focused on algebraic/symbolic representations and their manipulations. Students were enrolled in Algebra I (N=44), Geometry (N=19), Algebra II (N=65), Precalculus (N=32), and Calculus (N=18). Six problems were designed, and each student was given a problem to solve. Thus, each problem was solved by approximately one sixth of the participants. For example, one problem was finding a solution to a linear equation of two variables, given the graphic representation with two lattice points, $(0,-2)$ and $(-2, 4)$, and the incomplete algebraic representation $?x + 3y = -6$ where the coefficient of x is missing. Students were asked to give two solution strategies to solve the problem, one algebraic and the other graphic, using the symbolic equation of the line, and the graph of the line, respectively, and to explain their solutions. More than three fourths of students used an algebraic approach, even when the graphic approach required fewer solution steps. In their algebraic solutions, students manipulated the equation $ax + by = c$, and re-wrote it in slope-intercept form, $y = mx + b$. Then, using points from the line, they calculated the slope of the line, and

identified the missing coefficient in the equation. Students failed to see those points (x_0, y_0) as solutions to the linear equation $ax + by = c$. Fewer than one third of students chose a graphic approach, either as their primary or secondary solution. Knuth hypothesized that the Cartesian connection from the graph of the line to the symbolic equation of the line is more difficult to master. Knuth argued that students' reliance on symbolic representations is due to the curriculum and instructional approach that emphasizes symbolic manipulations, but also to the nature of the problems themselves, that required solutions to equations. As Arcavi (2003) points out, the graphic approach may have been rejected by students (and their teachers) because graphic representations are not precise, and could introduce estimates and inaccuracies.

In Knuth's (2000) study, student difficulties with the Cartesian connection were inferred by the researcher from student answers, without any interview input from students or their teachers. Also, all students experienced the same curriculum, raising the question of whether students experiencing a different curriculum might encounter the same difficulties. In the present study, students and teachers were asked to comment on difficulties. Furthermore, students' instructional programs varied by district.

Cheng (2010) studied the relationship between students' understanding of steepness and their abilities to reason proportionally. Participants were 413 students (156 students in Grade 6, 115 students in Grade 7, and 146 students in Grade 8) from the same public school, enrolled in either general mathematics (primarily Grade 6), or Pre-Algebra, Algebra I, or Geometry, and in one of three

tracks, regular, honors, or accelerated. Two tests were administered to all students, one focusing on ratio and proportionality, and the other on steepness. The ratio and proportionality test had tasks of varying levels of complexity. For example, in a task of lesser difficulty, a recipe for a soup for eight persons was given, with the necessary quantities for each of the ingredients. Students had to figure out quantities when the recipe was altered to four persons or six persons. In another task, of greater difficulty, three doctors used an office two, four, and six days per week, respectively. Together they paid $240 for lighting. Students had to figure out the amount each doctor needed to pay so that the bill would be "fair." The steepness test contained tasks of comparing two lines, two roofs, and two staircases. All lines, roofs, and staircases were represented graphically using homogenous systems of coordinates. Interviews were conducted with 16 of the middle school students who were enrolled in two private schools.

Cheng (2010) found a positive correlation between students' abilities to reason proportionally and their abilities to solve steepness problems. About a quarter of the variability in students' scores on the steepness test could be explained by their performance on the ratio and proportionality test. On the steepness test, students performed better in mathematical situations when asked to compare the slopes of two lines (80% correct answers), than in physical contexts when they were asked to compare the slopes of two roofs (66% correct answers) or two staircases (56% correct answers). In short, the difficulty of the task increased as the complexity of the situation increased i.e., more lines, angles and triangles appeared in the roofs and staircase situations than in the non-contextual lines situations. This finding is

not surprising, confirming previous research on proportionality/linearity (Kaput & Maxwell-West, 1994; Lamon, 1993; Lobato & Siebert, 2002).

Apart from the shortcomings recognized by the researcher herself, like those referring to the construct validity of the steepness test and the small sample of the interviewed students, there are also issues in terms of the statistical methods used in Cheng's (2010) study. The relationship between students' reasoning abilities about steepness, and their reasoning abilities about ratio and proportionality was determined using a two-factor ANOVA and linear regression. However, the assumption of independence necessary in traditional statistical analyses was violated, given that all students were from the same school, and students in the same class were taught by the same teacher. Thus, students shared the same curriculum and the same instructional approaches, and the observations within groups were not independent. The violation of the assumption of independence casts doubts upon any statistical significance reported in the study (Raudenbush & Bryk, 2002). Also, the researcher did not take into consideration natural clusters, like clustering by track or performance level (regular, accelerated or honors) or by current mathematics course (e.g., Regular Mathematics for 6th, 7th, or 8th grade, Pre-Algebra, Algebra I, Geometry). In short, the relationship between steepness and proportionality may be different for various groups of students (e.g., low-achieving vs. high-achieving, Pre-Algebra students vs. Geometry students). In the present study, analyses of student performance on the MDT are made by grade, current mathematics course, and performance/track level.

Stump (2001a) studied student understanding of slope or rate of change. Twenty-two high school students enrolled in Precalculus used bicycles to investigate the linear relationship between the number of pedal revolutions (independent variable) and the number of wheel revolutions (dependent variable) for various gears. Students collected data, and constructed scatter plots and the best-fitting line. Two months later, a test was administered and interviews were conducted with all 22 students. In the test, students were presented with a scatter plot, a graph of the best fitting line, and its symbolic representation $y = 2.24x + 0.101$, and were asked about the slope of the line. Only eight students mentioned a ratio or a rate and referred to the number of pedal revolutions and the number of wheel revolutions. When asked, "What is slope?" fifteen students used the word "angle," nine students mentioned "rise over run," four students referred to a formula (two students used the correct formula $\frac{y_2 - y_1}{x_2 - x_1}$, and two students reversed the ratio $\frac{x_2 - x_1}{y_2 - y_1}$), and eight students mentioned "steepness." When presented with a model of a ski ramp and asked to measure its steepness, only five students used a ratio (three students used "rise over run," and two students used the formula $\frac{y_2 - y_1}{x_2 - x_1}$). Five other students thought of the height and the length of the ramp, but did not know how to use those to determine the slope. In another task, the graph of a linear function in a non-homogeneous system of coordinates was presented (independent variable - number of tickets sold for a dance concert, and dependent variable- profit in dollars) and students were asked to figure out

the cost of a ticket and the slope of the graph. Only seven students correctly identified the cost of a ticket, and only three of the seven students correctly identified the slope of the line. Seven students stated that they did not know how to find the cost of the ticket, and six students answered that they did not know how to find the slope of the line. The remaining students gave incorrect answers. All but four students correctly calculated the rate of growth given two sets of values (age, height) for a child, $(9, 4'3'')$ and $(11, 4'9'')$. In general, Stump's students performed better when asked to find the rate of change in functional situations (e.g., finding the cost of a ticket given the number of tickets sold and the profit) than when asked to find the slope as a measure of steepness (e.g., finding the slope of a wheelchair ramp).

The small number of participants makes it difficult to generalize the findings of Stump's (2001a) study to other populations of students.

Teuscher and Reys (2010) studied student difficulties with identifying the rate of change from a graph. Participants were 191 high school students enrolled in Advanced Placement (AP) Calculus. The graph of a piece-wise function was presented (similar to MDT problem 7, Appendix A) and the task was to identify the interval where the rate of change is 2. About 37% of students failed to identify the interval. When asked to identify the interval on which the function had the greatest rate of change, 57% of students disregarded the sign of the rate of change, positive $(+2)$ or negative (-2), and took into consideration the absolute value, 2. Teuscher and Reys believed that student difficulties in finding the greatest rate of change stemmed from difficulties with vocabulary, the meaning

assigned by students to the phrase, "greatest slope." There is the possibility that, in spite of instruction, students associated slope with the steepness of the hill or mountain and the difficulty of climbing the hill, and disregarded the orientation of the hill i.e., did not notice whether positive changes in the vertical height (change in *y*) were associated with positive changes in the horizontal length (change in *x*).

Teuscher and Reys (2010) did not take into consideration the complexity of the problem or students' lack of familiarity with the situation (a graphic representation of a piece-wise function), that could have played a role in students' failure to correctly solve the tasks. Student difficulties were inferred by researchers from student solutions, with no input from students or their teachers in the form of interviews.

Orton (1983) studied student understanding of various concepts of Calculus, including rate of change. Reported here is only the part of the study focusing on constant rate of change. Orton conducted interviews with 60 students in high school, ages 16-18 from four high schools, and 50 students, ages 18-22, from two colleges. The college students were preparing to become mathematics teachers. One of the tasks concerned rate of change of a linear function in a real-life situation (water flowing in a tank at a constant rate of 2 units of depth per 1 unit of time) and was accompanied by the corresponding table of values, with the values 0, 1, 2, 3, 4, 5 for the independent variable *x*, and the values 0, 2, 4, 6, 8, 10 for the dependent variable *y*, and the graph of the function, $y = 2x$. The task required finding the rate of change at $x = 2\frac{1}{2}$ units of time, and at $x = T$, an

arbitraty number of units of time. For the second task, concerning the rate of change of a linear function, the context was mathematical. The graph and the symbolic representation of the linear function $y = 3x - 1$ were presented, and participants had to figure out the rate of change at $x = 2\frac{1}{2}$ and at $x = T$. Both graphs used a non-homogeneous system of coordinates. Both tasks were scored as one item using a five-point scale {0, 1, 2, 3, 4}. The mean score for the high school students was 2.22 , and for the future teachers, 2.02. For the first task, 20% of students answered that the rate of change at $x = 2\frac{1}{2}$ is 5 (the value of the function or the *y*-coordinate corresponding to $x = 2\frac{1}{2}$), and more than 20% of students failed to respond or gave the answer "*2T*" for the rate of change at $x = T$. On the second task, with the graph of the linear function and no real-life context or table of values, 26% of students attempted or indentified the rate of change at $x = 2\frac{1}{2}$ as the value of the function $y = 3x - 1$or the *y*-coordinate corresponding to $x = 2\frac{1}{2}$, and gave the answer "$3x - 1$" for the rate of change at $x = T$.

Interviews of all students about their solutions revealed that incorrect answers were due to incorrect understanding of rate of change, and to weak understanding of proportionality, but not to computational errors. For example, students did not know that regardless of the "slope triangle" used, the same ratio for the slope of the line would result.

Orton's (1983) study demonstrated that high school and future mathematics teachers alike do not have a comprehensive understanding of the concept of rate of change. Difficulties with rate of change occurred across both groups, regardless of age and mathematical experience.

Zaslavsky et al. (2002) investigated 124 participants' understanding of slope in non-homogenous systems of coordinates, that is, whether slope is an analytical attribute of the linear function represented by a line or a geometric attribute of the line. Participants included 28 high school students in Grade 11 enrolled in Calculus, 28 pre-service secondary mathematics teachers, 50 experienced in-service mathematics teachers, 14 mathematics educators involved in research, and four research mathematicians. Two tasks were administered to all participants, and interviews were conducted with 12 of the participants. In Task 1, the function $f(x) = x$ was graphically represented in a homogenous system of coordinates, and participants were required to find the slope of the function, and the tangent of the angle β between the line and the positive sense of the *X*-axis. In Task 2, the function $f(x) = x$ was graphically represented in a non-homogenous system of coordinates, and participants were also required to find the slope of the function, and the tangent of the angle β between the line and the positive sense of the *X*-axis. Participants' answers revealed three perspectives on slope: analytical, geometric, and mixed. Those perspectives were common across all groups, high school students, pre-service teachers, experienced teachers, mathematics educators, and mathematicians. Twenty-four participants held an analytical perspective of slope, i.e., slope is a property of the linear function and does not

depend on the graphic representation, and found $m = 1$ and $\tan\beta = 1$. Ninety-one participants held a mixed perspective of slope, i.e., they considered that the slope of the function $f(x) = x$ is $m = 1$, but 51 participants could not find a value for $\tan\beta$ or dismissed the question, while 40 participants calculated $\tan\beta \neq 1$ ($\tan\beta = \frac{1}{3}$). Eight participants held a geometric or "visual" perspective of slope, i.e., slope is an attribute of the line that varies when the system of coordinates changes, and answered $m = \frac{1}{3}$ and $\tan\beta = \frac{1}{3}$. To illustrate those perspectives, five participants' (three mathematicians and two doctoral students in mathematics education) responses were reported in detail. Participants holding either an analytical or a geometric perspective of slope did not encounter cognitive conflicts, whereas participants with a mixed perspective had difficulties reconciling the two perspectives. The conflict was generated by the conviction that $m = \tan\beta$, which was contradicted by the "visual" appearances when a non-homogeneous system of coordinates was used.

Although the researchers reported in detail on the slope perspectives held by only the Ph.D. candidates in mathematics education and holders of Ph.D. degrees in mathematics, those perspectives were common across all groups of participants.

2.4.2 Teaching Experiments on Student Learning of Linearity and Linear Functions. In teaching experiments, the focus is primarily on student learning trajectories. The review of teaching experiments on student learning of linearity

and linear functions emphasizes student difficulties and researchers' recommendations for curriculum and instruction.

Lobato and Siebert (2002) conducted a 10-day teaching experiment with nine students in Grades 8-10 and analyzed the students' abilities to create a "ratio-as-measure" for slope. Tasks required students to decide what measurements were needed to assess the steepness of a ramp, determine the steepness of the ramp, and modify the ramp such that its steepness remains the same but its height increases. During interviews, students were asked, "What is slope?" and were invited to comment on other students' ideas about measuring the steepness of ramps. They also were asked to draw a picture of a ramp with a slope of 2. Only one student's conceptual development was reported, the student who made the most progress during the teaching experiment. That student was at the end of the Grade 8 year, and had just completed Algebra I. Although the student was acquainted with the slope formula, at the beginning of the teaching experiment he attended only to the height of the ramp as a measure of steepness, and treated the width of the ramp as resulting from changes in height. Gradually, he coordinated the height with the width of the ramp and considered them as two quantities changing in tandem, and with a constant ratio.

In another situation in the Lobato and Siebert's (2002) study, involving a software simulation of a linear movement of a clown and a frog, students had the task of assigning values for distance and time so that the frog's walking speed matched that of 10 cm in 4 seconds. In summary, the student who successfully created a ratio as measure for the steepness of a wheelchair ramp, also

successfully solved the task of finding values for distance and time (e.g., 2.5 cm in 1 second) to match the frog's speed with the clown's speed. However, he failed to see how the situations were alike mathematically. Lobato and Siebert pointed to student difficulty transferring between different situations or contexts.

The complexity of the situations should be considered in the interpretations of the results of these studies. For example, the situation involving slope as a measure of the steepness of a ramp is more complex than the situation involving linear moment with a constant speed (Lobato & Siebert, 2002). Reasoning proportionally is most difficult for students when the geometric aspects are more complex (Kaput & Maxwell-West, 1994; Lamon, 1993). The ramp situation involves proportionality resulting from similarities of triangles. By contrast, the linear movement context involves direct variation of distance and time.

Hines (2002) conducted a teaching experiment with one Grade 8 student to gain insight into student's understanding of linear functions and connections among representations of linear functions. Hines used a dynamic physical model as a concrete way of representing functions as covariational processes involving two quantities. One model of a linear function was a spool elevating system with the independent variable being the number of rotations of the handle on the spool system, and the dependent variable being the height of the object. The development of the student's knowledge of rate of change was similar to the one described by Thompson (1994b). At the beginning, the student had an action view of functions, concentrating on the input and output values, one a time. At the end of the teaching experiment, the student held a process view (Sfard, 1991; Slavit,

1997) of linear functions, with the dependent variable and the independent variable varying simultaneously. Moreover, the student was able to make connections among various representations of linear functions (tables, graphs, equations). Since Hines created special learning experiences for the student, generalizing the findings to other populations of students is difficult.

A teaching experiment was performed by Ellis (2009) with two groups of students. Group 1, consisting of seven Grade 8 Algebra I students studied linearity and linear functions using a number-pattern approach over a period of 12 lessons, each one hour and 50 minutes in duration. Students were presented with a table of values, and were asked to identify the pattern and continue it by adding new entries into the table. Group 2, consisting of seven Grade 7 students, studied linearity and linear functions with a focus on quantitative relationships, over a period of 15 one-hour lessons. During instruction, students were presented with two-real world situations: 1) rotations of physical gears, and 2) linear movement with constant speed simulated by computer software and involving two characters. In one of the tasks, a table of values was given with distances and times representing the movement of a frog (one of the software's characters) and students were asked if the frog was going at the same speed all of the time, speeding up or slowing down. Ellis found that Group 2 students, who used quantitative relationships, could more easily make ratios from quantities and generalize correctly about linear functions, than did Group 1 students who focused on number-patterns.

Ellis‘s (2009) study points to the importance of quantitative reasoning for student understanding of slope or rate of change, in agreement with Lobato and Siebert (2002).

Davis (2007) studied student understanding of the *y*-intercept of linear functions in real-world contexts and across multiple-representations, and the terminology students used to describe relationships. Participants were eight high school students (two students in Grade 10, three students in Grade 11, and two students in Grade 12), enrolled in Algebra II, and their teacher. Students and their teacher were observed during 32 lessons over a ten-week period of time. A pre-test was given to assess students' abilities to make connections between different representations. During instruction, students were allowed to invent their own terminology for the *y*-intercept, and they used "starting point." The term "starting point" could be explained by the fact that most of the real-life situations used during instruction were modeled with linear functions that had time as the independent variable (e.g., paying an amount of money over a period of time; the cost of a phone call of different durations). For the post-test, participants had to define the "starting point" and translate between different representations of the "starting point." Interviews were conducted with all participants. Analyses of observations and interviews revealed that students' successful use of the "starting point," suggested by the real-world contexts, did not translate into successful use of the *y*-intercept when symbolic, tabular, and graphical representations were involved. Moreover, the use of the informal terminology seemed to have a negative impact on student performance on abstract tasks, i.e., tasks without real-

world contexts. Of note, the teacher did not question his assumptions about students' meaning of the "starting point," and was unaware of the influence of the instructional materials upon his students' meanings as well as students' difficulties to translate their knowledge of the "starting point" between contexts and representations.

Although conducted with a small number of participants, the observations lasted for a long period of time, in a real classroom and with both students and their teacher. Davis's results confirmed those of other studies showing that students have difficulty transferring knowledge of the *y*-intercept between situations (Schoenfeld et al., 1993; Smith et al., 1989).

The Functions Group at Berkeley conducted a teaching experiment of seven hours duration over a period of seven weeks, with a 16-year-old student to explore the student's change in knowledge of linear functions, their representations, and connections between representations, particularly the Cartesian connection (Schoenfeld et al., 1993). The student was concurrently enrolled in a Calculus summer course at The University of California-Berkeley. In Grade 8, the student completed Algebra I on her own, with little help from a teacher.

Schoenfeld et al. (1993) conducted the teaching experiment using the software environment Grapher, designed by the Functions Group at Berkeley, and containing three microworlds. The software allowed the user to input a function in symbolic form (e.g., $y = 2x + 5$) and *x*-values. For each *x*-value, the software calculated the corresponding *y*-value, explained why the pair (*x*, *y*) would be on the graph of the function, entered the coordinate pair into a table of values, and

plotted the point on a graph. The software also allowed explorations of the roles of *m* and *b* as parameters. The student assigned various values to *m* in $y = mx + b$ and the software accordingly rotated the graph of the function through the fixed *y*-intercept. The student also assigned various values to *b* in $y = mx + b$ and the software accordingly moved up or down the graph of the function. Finally, the student could play the Targets game on a Cartesian system of coordinates. In the game of Targets, lines appear and the student had to "shoot" them by entering the symbolic form of the equation of the graphed line. The student explored all of the software's tasks, but spent most of the time playing the game of matching a line graphed in a Cartesian system of coordinates to its symbolic representation. A researcher observed the student interacting with the software and asked questions about her understanding. Analysis of student answers showed that the student changed her conceptual understanding about the necessity of three descriptors to graph the line $y = mx + b$ (slope *m*, *y*-intercept, and *x*-intercept) to two descriptors (slope *m*, and *y*-intercept). However, there was limited and unstable change in the student's knowledge of the connection between the symbolic representation of the function $y = mx + b$, its coefficients *m* and *b*, and their graphic representation.

At the beginning of the teaching experiment conducted by Schoenfeld et al. (1993), the student's knowledge of slope was limited to the algorithm of calculating the slope using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, with no connection to the graphic representation. She did not know that $y_2 - y_1$ and $x_2 - x_1$ have magnitude and sense (positive or negative) specified by mathematical

conventions, and could not coordinate the negative sign of $y_2 - y_1$ with a decrease in the *y*-value, or the positive sign of $x_2 - x_1$ with an increase in the *x*-value. Gradually, during the teaching experiment, she could relate the sign of *m* to the orientation of the line and the magnitude of $|m|$ with the inclination of the line, but without justifying it with the Cartesian connection or having any basis for her rule. Until the end of the experiment, the slope formula remained a "black box" for the student. Additionally, although at the beginning of the experiment she knew that *b* was called the *y*-intercept, she did not connect it with the fact that the point (0, *b*) had to be situated on the line.

Although only one student's difficulties were reported, Schoenfeld et al. (1993) mentioned that these difficulties were shared by other participants in the study. The student's lack of connection between the slope formula and the change in *y* and the change in *x* represented as line segments with oriented magnitudes is seen by Schoenfeld et al. as an example of missing the Cartesian connection.

In another teaching experiment by the Functions Group at Berkeley, with two students in Grade 8, and with the same software, Grapher, Moschovich et al. (1993) focused on student flexibility to move between the process-view of functions and the object view of linear functions. During the teaching experiment students had to complete various tasks. For example:

1) Given a table of values, students had to write the symbolic representation of the linear function (process-view).

2) Given a graph of a group of lines with the same *y*-intercept (or with the same slope), students had to write the equation of one of the lines (object-view).

Other more complex tasks required both a process-view and an object-view of linear functions. For example, given a line and a point not situated on the line, students had to write the equation of the line parallel to the given line and passing through a given point.

Both students had difficulties moving flexibly between the process and object perspectives of linear functions. One of the students, who had not been previously exposed to a linear functions curriculum, also had difficulties moving within the process and object perspectives of linear functions. The researchers explained the students' difficulties with connecting between various function representations by the fact that the conceptual development of functions as processes-objects might not be a linear one - from process to object with the object view reserved for the mathematicians - but rather a simultaneous one. Both perspectives, and the flexibility of moving between perspectives, are necessary to solve complex problems.

2.4.3 Studies on Student Understanding of Slope or Rate of Change in Other Content Areas. Research on student understanding of slope or rate of change in other content areas focuses on students' abilities to connect various representations of real-world phenomena, and their flexibility to transfer knowledge between mathematical and real-life situations.

The Physics Education Group at the University of Washington investigated student difficulties connecting graphs with speed in real-world situations (McDermott et al., 1987). This multi-year study was based on observations of several hundred participants, including college students enrolled in a preparatory course for Physics, pre-service and in-service teachers enrolled in a university Physics course, college students enrolled in Honors Calculus-based Physics at the university, and high school students enrolled in Physics classes at local high schools. Although the degree of severity varied, student difficulties were common across all groups. For example, in one of the tasks, a graph (position versus time) representing the linear motion with constant speed of two objects was presented in a non-homogeneous system of coordinates. Participants were asked to compare the speeds of the two objects at the moment $t = 2$. Many of the participants who incorrectly compared the two speeds focused on the change in position at $t = 2$, and decided that the higher vertical-coordinate (greater height of position) indicated the object with the greater speed. Of great interest is that the same conceptual error was noticed in another task, when the graph position versus time of an object in motion was given, and the participants were asked to sketch the graph of speed versus time. Researchers hypothesized that most of the student errors and difficulties were due to the participants' inability to interpret graphs, i.e., to connect between the graphic representation and the situation it represents.

Newburgh (2001) conducted a teaching experiment with twenty-six Grade 12 high school students enrolled in Physics to investigate their understanding of slope in one situation and their abilities to transfer their knowledge of slope to

other situations. Students measured the areas and the heights of various rectangles, all with the same width. Then, they completed a table of values (independent variable - height, dependent variable - area), and constructed scatter plots and the best-fitting line. When asked for the numerical value and the meaning of "slope of the line," only six students identified it as the width of the rectangle. The majority of students omitted the units of measurement when referring to slope. The same task was repeated in different situations (e.g., the movement of a cart with constant acceleration in which students measured velocity and time). The students who participated in this teaching experiment were more likely than contrast-group students to transfer their knowledge of slope to other situations. Newburgh's findings, i.e., the flexibility of transfer between situations, are not in concert with the findings from other studies, as for example Lobato and Siebert (2002).

2.5 Research on Teacher Knowledge of Student Understanding of Linearity and Linear Functions

The studies conducted by Orton (1983), McDermott et al. (1987), and Zaslavsky et al. (2002) cited previously, showed that pre-service and in-service teachers have difficulty understanding linearity and linear functions. Some teaching experiments (Davis, 2007; Thompson & Thompson, 1994) revealed that teachers are not always successful in attending to their students' thinking. For example, in Davis's teaching experiment on student understanding of the *y*-intercept (Section 2.4.2), the teacher did not recognize his students' difficulties

connecting various representations of the *y*-intercept. He assumed that his students could move flexibly between various representations, contexts and situations in which *y*-intercept occurred.

Several additional studies on teacher mathematical and pedagogical knowledge of slope or rate of change, and connections between representations are described below. The pedagogical knowledge is focused on understanding of student difficulties.

In a study conducted with in-service (N=21) and pre-service (N=18) mathematics teachers, Stump (1997) investigated their knowledge of various representations of slope and their pedagogical knowledge for teaching the concept of slope. All but two of the in-service teachers held Bachelor degrees in mathematics or mathematics education, and eight of the 15 teachers with Master degrees had a Master in mathematics or mathematics education. All pre-service teachers planned to teach at the high school level, and all but one had teaching experience as part of their training prior to the study. An initial survey was administered to all participants. Additionally, from each of the in-service and the pre-service teacher groups, eight teachers, four females and four males, were randomly selected for interviews. The initial survey and interviews included problems and questions about teachers' mathematical and pedagogical knowledge of slope.

Analyses of teachers' surveys and interviews showed that, on average, teachers mentioned two representations of slope. More than 90% of the pre-service teachers and all of the in-service teachers mentioned slope in geometric-analytical

context, i.e., slope as "rise over run" or slope as change in *y* over change in *x*. The next most often mentioned context was slope as rate of change. More pre-service teachers (about 50%) mentioned slope as rate of change than did in-service teachers (about 40%). Almost 40% of the in-service teachers, but only about 15% of the pre-service teachers, mentioned slope in physical contexts, as, for example, a measure of steepness of hills, ski slopes or wheelchair ramps. Slope as the parameter *m* in $y = mx + b$ was mentioned by about one third of the pre-service teachers, and by only one in-service teacher. None of the pre-service teachers, and only one of the in-service teachers mentioned slope in a trigonometric context, as the tangent of the angle of inclination. Slope as a ratio was mentioned by about the same percentage (20%) of in-service and pre-service teachers.

All teachers correctly calculated the rate of growth of a child height given two sets of values (age, height) for the child, $(9, 4'3'')$ and $(11, 4'9'')$ (Stump, 1997). Also, all teachers correctly calculated the slope given two sets of values (duration, cost) for a phone call, (5 minutes, $5.30), (10 minutes, $10.80), and interpreted it as rate of change. However, two of the in-service teachers considered that the cost of the phone call per minute was not constant, it increased with time. Both in-service and pre-service teachers had difficulty with slope as rate of change in situations requiring connections between graphic representations and linear movement with constant speed. For example, in one of the problems taken from McDermott et al. (1987), a graph (position versus time) representing linear motion with constant speed of two objects ($m_1 \neq m_2$) is presented in a non-

homogeneous system of coordinates. Teachers were asked to compare the speeds of the two objects at the moment $t = 2$. Two thirds (67%) of the in-service teachers and 71% of the pre-service teachers correctly solved the problem. Four pre-service teachers and three in-service teachers considered that the higher vertical-coordinate (greater height of position) indicated the object with the greater speed. Although $m_1 \neq m_2$, and therefore the speeds are different, two pre-service teachers and one in-service teacher considered that the two objects could have the same speed at some moment in time. Two in-service teachers did not respond. One first-year in-service teacher mentioned that the idea of slope as rate of change was new to her. These findings are supported by the work of McDermott et al. (1997).

All teachers could write the linear function (x = number of hours, y = number of points for score) for the following situation: A student initially gets 50 points on a test, and for each hour of study, her score increases by 3 points. However, teachers had difficulty identifying the slope and the y-intercept in linear functions in the symbolic form $y = mx + b$, when m and b were not numbers. In one of the problems, the formula for the expansion of the length of a metal rod under changes of temperature is given, $l - l_0 = al_0(t - t_0)$, where l is the length of the rod, t is the temperature, l_0 is the initial length, t_0 is the initial temperature, and a is a constant for the type of metal. The task was to express the length of the rod as a linear function of temperature, $l(t) = mt + b$, and to identify the slope and the y-intercept. About the same percentage of pre-service teachers (83%) as in-service

teachers (81%) correctly wrote the formula in linear function form (or slope-intercept form). However, less than two thirds of them (61% of the pre-service teachers and 62% of the in-service teachers) correctly identified the slope. Only 33% of the pre-service teachers and 57% of the in-service teachers correctly identified the *y*-intercept. Two pre-service teachers and three in-service teachers did not respond.

Only one-third (33%) of the pre-service teachers and two-thirds (67%) of the in-service teachers correctly calculated the slope of a line given a point on the line, and the measure of the angle of the line with the positive sense of *X*-axis. The teachers who could not calculate the slope of the line in a trigonometric context often referred to the need for a second point on the line.

Stump's (1997) interviews with teachers confirmed the survey findings. Both in-service and pre-service teachers had difficulty recognizing slope as a parameter, as constant rate of change of two covarying quantities, and interpreting graphs that required connecting various representations of slope. The in-service teachers identified more connections for slope representations than did the pre-service teachers. For example, the in-service teachers could connect between the geometric-analytical context of slope and the trigonometric context of slope.

Teachers' weak content knowledge of slope was mirrored by their weak pedagogical knowledge. When asked about their mathematical knowledge, the most often mentioned context for slope was the geometric-analytical one. However, analyses of teachers' pedagogical knowledge revealed that they most often mentioned the physical context of slope for instruction. Their instructional

practices, as mentioned during surveys and interviews, reflected only slope in physical and geometric-analytical contexts, and lacked other contexts for slope as well as connections between various contexts/representations of slope. Stump (1997) pointed out that teachers did not question their students' abilities to make connections between various representations or transfer among them. When asked to identify student difficulties with slope, the in-service teachers mentioned difficulties with graphing lines and computing slope, while the pre-service teachers mentioned difficulties with understanding what slope represents. Stump recommended more research to study the impact of the mathematics curriculum on teachers' weak understanding of slope, and the effect of teachers' weak mathematical and pedagogical knowledge of slope on their students' understanding of slope.

Stump's (1997) study of teachers' mathematical knowledge of slope was comprehensive, and the categories used for slope contexts/representations were used for the categories of conceptual knowledge of slope in the present study (see Chapter 3).

Stump (2001b) conducted another study on the development of future mathematics teachers' knowledge of representations for teaching slope, and knowledge of students' difficulties with slope. Participants were six future secondary mathematics teachers enrolled in a one-semester methods course. Prior to the start of the methods course, a survey was administered to gauge future mathematics teachers' knowledge of slope and knowledge of student difficulties with slope. The methods course included activities and class discussions on

various contexts and representations for slope (e.g., geometric, analytical), and emphasized slope as rate of change in functional situations. Assignments included interviews with one high school student and one college student about their difficulties with slope, interview analyses, and an algebra textbook analysis that focused on the topic of slope, as well as the preparation of lesson plans for teaching the concept of slope to middle school and high school students. After the course completion, five of those participants taught a basic algebra course at the university. Lessons on slope taught to college students during the basic algebra course were videotaped and analyzed. To account for the development of future teachers' knowledge of representations for teaching slope and knowledge of students' difficulties with slope, future teachers' prior knowledge was compared with their knowledge developed during the methods course, and with the knowledge demonstrated during the actual teaching.

At the beginning of Stump's (2001b) study, the future mathematics teachers referred to slope as a measure of steepness in graphic representations (e.g., the slope of the line $y = x$) and physical situations (e.g., inclined plane, roof). At the end of the study, their representations of slope expanded to functional situations (slope as rate of change), and graphic representations of lines and their symbolic equations. Results for only three future teachers are reported. One future teacher remained focused on physical situations (slope as a measure of steepness), another one emphasized the functional situations (slope as rate of change), while the third used both physical and functional situations. However, only one of the future mathematics teachers used real-life situations in problems, while the other two

referred to real-life situations only as illustrations. In short, teachers differed in their emphasis on slope as rate of change and their use of real-life situations in problems. Stump hypothesized that the limited use of real-life and functional situations might be due to the influence of the traditional curriculum of the algebra textbook used in the basic Algebra I course.

With respect to understanding of student difficulties with the concept of slope, one of the future teachers thought that student difficulties with slope stemmed from weak procedural knowledge (e.g., not knowing the formula for slope, how to apply it or how to correctly perform calculations). During the two interviews with students, he discovered that students had conceptual difficulties. Consequently, he developed lessons to emphasize the conceptual understanding of slope. Another teacher, who initially thought that students would know the meaning of slope, discovered during the interviews that students did not know what slope was or how to calculate it. She decided to develop lessons that introduced the concept of slope as a measure of steepness. Interviews with students confirmed the third teacher's initial hypothesis that students have a weak understanding of slope as a measure of steepness. Accordingly, she developed lessons focused on slope as a measure of steepness ("slant"). In conclusion, the future teachers' initial assumptions of student difficulties with slope were changed or confirmed by interviews, and then used in the development of their lessons.

Stump's (2001b) study demonstrates the importance of teachers' content and pedagogical knowledge of various representations of slope and of student difficulties with slope. Only one of the future teachers' assumptions was

confirmed by student interviews, and reflected only one aspect of slope as a measure of steepness. Two of the future teachers' assumptions about students' understanding of slope were contradicted and had to be changed.

Even and Tirosh (1995) studied 162 prospective secondary mathematics teachers' subject-matter knowledge and knowledge about students' understanding of functions and undefined mathematical operations. All participants were in the final stage of their pre-service preparation. An open-ended questionnaire was completed by 152 of the prospective teachers, and interviews were conducted with the other 10 prospective teachers. The questionnaires and the interviews included problems and student solutions to problems about functions and undefined mathematical operations. Reported here are only the aspects of teachers' pedagogical knowledge referring to student understanding of linear functions. Teachers were asked to respond to a student error about linear functions, namely that the slope of the line representing the function varies proportionally with the measure of the angle between the line and the positive sense of the *X*-axis.

In one problem, a point *A* is shown in a rectangular system of coordinates, but neither units of measurements are given, nor are coordinates marked. Should the units of measurement be marked (assuming a homogeneous system of coordinates), then the coordinates of *A* may be (1, 9) meaning that the point *A* is in the first quadrant, very close to the *Y*-axis. The task was to figure out the equation of the line passing through the point *A* and the origin *O* of the system of coordinates. A student's answer accompanied the problem:

> Well, I can use the line $y = x$ as a reference line. The slope of line *AO* should be twice the slope of the line $y = x$, which is 1. So, the slope of line *AO* is about 2, and the equation is about $y = 2x$, let's say $y = 1.9x$. (Even & Tirosh, 1995, p. 13)

The prospective teachers were asked to comment about what the student had in mind when she responded, and if her answer was correct. Only half of the teachers correctly identified the student error and could describe the student's way of thinking: Since the line $y = x$ has the slope 1 and makes a 45° angle with the positive sense of the *X*-axis, and *AO* makes almost a 90° angle with the positive sense of the *X*-axis, almost twice as much as 45°, then the slope of *AO* should be almost twice the slope of the line $y = x$. Interestingly, some of the teachers who noticed the student's connection between the magnitude of slope and the measure of the angle of inclination, could not correctly solve the problem. They considered that the line *AO*, being so close to a vertical line, ought to have a large slope (e.g., 100). Some counted "rise over run," and found a slope of 10-15, which contradicted their idea that the slope ought to be a very large number. The other teachers, who did not notice the student's connection between slope and angle, considered that the student's error was due to an incorrect estimation of slope.

Even and Tirosh (1995) pointed out that the relationship between mathematical knowledge and knowledge about student understanding is a complex one, and both types of knowledge are necessary for teaching. The researchers found that only half of the prospective secondary mathematics teachers could attend to student reasoning about linear functions in the problem described above, and

some of those teachers could not solve the problem themselves. The researchers' findings confirm Stump's (1997, 2001b) findings about teachers' weak mathematical knowledge of slope, and weak knowledge of student understanding and difficulties with the concept of slope.

Cunningham (2005) administered a survey to 28 algebra teachers to determine how often problems requiring connections between function representations are used during instruction and in teacher-made assessments. Since the teachers taught students in Grades 8-10, the survey contained only problems focusing on linear functions, appropriate for Algebra I curriculum. The 12 survey problems required making connections between algebraic/symbolic (A), numerical/tabular (N), and graphic (G) representations. Teachers were asked to identify the number of class periods and teacher-made assignments that included problems similar to those in the survey. For the problem requiring connecting graphic and numerical representation, G to N, a linear function graphed in a homogeneous system of coordinates is shown and the task was to find an ordered pair that satisfies the linear equation of the line. The teachers mentioned that they spent the least amount of time during instruction, and used the fewest number of problems of that type in their assessments of the connection G to N (mean 1.35 class periods, mean 1.22 used in assessments). For the problem requiring connecting between algebraic/symbolic and numerical/tabular representations, A to N, a linear equation of a line in standard form was given and the task was to complete a table of values. Teachers mentioned that they spent 1-2 class periods teaching the connection A to N (mean 1.71 class periods) and used 2-3 problems requiring the

connection A to N in their assessments (mean 2.45 used in assessments). For the problem requiring connecting between the graphic and algebraic/symbolic representations of a linear function, G to A, a linear function was graphed in a homogeneous system of coordinates and the task was to find its symbolic representation $y = mx + b$. Teachers mentioned that they spent 2-3 class periods teaching the connection G to A (mean 2.21) and used 2-3 problems requiring the connection G to A in their assessments (mean 2.67).

Cunningham (2005) noted that teachers gave so little attention to those connections because they assumed that their students would not encounter difficulties making the connections. However, studies cited previously identify student difficulties with the Cartesian connection (Smith et al., 1989; Schoenfeld et al., 1993; Moschkovich et al., 1993; Knuth, 2000). Cunningham also pointed out that algebra teachers assume that their students will not have difficulties connecting between various representations of functions, especially from graphic to numerical representations. As a result, they allocate the least amount of instructional time to connections among representations, and rarely assess student understanding of these connections.

You (2006) studied teachers' mathematical and pedagogical knowledge of linear functions with a focus on various representations of those functions. Reported here is only the part of the study focusing on teachers' mathematical knowledge of the Cartesian connection and the relationship to their abilities to recognize student difficulties with connections among representations. Participants were 104 elementary and middle school pre-service teachers in the

last stage of their preparation and enrolled in methods of teaching elementary/middle mathematics courses. One of the problems presented a table of values representing the value (in dollars) of a car after a numbers of years since purchase: (0, 16800), (2, 13600), (4, 10400), (6, 7200). Teachers were asked to write the linear function representing the depreciation of the car's value over time. Ninety-eight teachers did not write the correct linear function. Thirty-two teachers did not properly attend to the change in time, i.e., did not notice that the time interval was 2 years, and wrote $V(t) = -3200t + 16800$. In another problem, taken from Knuth's (2000) study (Section 2.4.1), only 27 teachers gave the correct answer, and 20 used an algebraic approach. The problem required finding the equation of a line, given the graphic representation with two lattice points, $(0,-2)$ and $(-2, 4)$, and an incomplete algebraic representation, $?x + 3y = 6$, where the coefficient of *x* is missing. You's findings about teachers' knowledge of the Cartesian connection confirmed Knuth's (2000) findings about student difficulties with the Cartesian connections, and their overreliance on algebraic solutions.

Moreover, You (2006) found that the teachers' knowledge of representations of linear functions is related to their pedagogical knowledge, particularly their ability to attend to their students' understanding of representations of linear functions. The pre-service teachers did not successfully recognize student errors, misconceptions, or difficulties with linear functions and their representations. You's (2006) study results also confirmed Stump's (1997, 2001b), Even and Tirosh's (1995), and Cunningham's (2005) findings. Teachers' weak mathematical knowledge is positively associated with weak pedagogical

knowledge, particularly teachers' abilities to attend to their students' understanding of linearity and linear functions.

2.6 Student and Teacher Identification of Problem Difficulty

Recognizing and identifying the relative degrees of problem difficulty with accuracy is critical to the construction of assessment instruments and to the design of instructional approaches that capitalize on what students know. Identifying problem difficulty is equally important for students, enabling them to identify their strengths and weaknesses and subsequently take greater responsibility for their own learning. The following research studies refer to measures of problem difficulty. In the discussion that follows, studies of student assessment of problem difficulty are presented first followed by studies comparing student and teacher assessment of problem difficulty, and then studies on teacher assessment of problem difficulty.

Lee and Heyworth (1996) conducted a study to develop a measure of problem difficulty. Participants were Grade 9 students (N=125) from two secondary schools in Hong Kong, and mathematics teachers (N=28) enrolled in mathematics education courses at university in Hong Kong. Two mathematics tests, with a total of 32 problems requiring the simplification of algebraic expressions with logarithms and the solving of logarithmic equations, were administered to all students. Students also were asked to rate the degree of difficulty of each problem on a scale from 1 to 5 (1-very easy, 5-very hard). Teachers were not asked to solve the problems. Instead, they were asked to assess the level of

difficulty of each problem and identify the factors that might influence the level of problem difficulty. Student perceptions of problem difficulty correlated highly with their performance on the two tests. Teacher assessments of problem difficulty for students correlated moderately low with student performance. Using a regression framework, Lee and Heyworth identified the factors predicting the three measures of problem difficulty given by: 1) student perceived difficulty, 2) student performance measured as a ratio between the number of correct answers and the total number of answers, and 3) teacher assessment of problem difficulty. Student perceived level of problem difficulty could be predicted by the following four factors: 1) perceived number of difficult steps required to solve the problem, 2) number of algorithmic steps necessary to solve the problem, 3) number of operations necessary to solve the problem, and 4) student familiarity with the type of problem (e.g., logarithmic equations that can be brought in the general form $\log_a f(x) = \log_a g(x)$). Results showed that student performance could be predicted only by the number of algorithmic steps necessary to solve a problem. Teacher assessment of problem difficulty could be predicted by student familiarity with the problem, the number of "log" occurrences in the problem, and the numerical complexity of the problem. The difference between the factors predicting student perception of problem difficulty and those factors predicting teacher assessment of problem difficulty might be explained by the fact that teachers were not asked to solve the problems. Consequently, teachers were more likely to judge the level of problem difficulty using surface features of the problem (e.g., number of "log" occurrences in the problem).

The researchers neither described the student sample nor accounted for the fact that student observations were not independent, as students came from the same two schools, and most likely, some of them were enrolled in the same mathematics classes, taught by the same teachers. Although the *p*-values reported are small, and the results about the factors proposed as predictors for different measures of problem difficulty are supported by previous research literature (Jerman, 1973; Silver & Thompson, 1984; Lane, 1991; Mason, Zollman, Bramble, & O'Brien, 1992), more research is needed to develop a measure for problem difficulty that accounts for student perceived difficulty. Nevertheless, the Lee and Hayworth's study is important because it points out that students are better able to predict their performance than are their mathematics teachers.

A study by Lingard, Minasian-Batmanian, Vella, Cathers, and Gonzalez (2009) was conducted to compare student and faculty perception of problem difficulty in first-year Biochemistry and Physics courses. Participants were Australian undergraduate students in their first year of college and enrolled in Biochemistry or Physics. About 400 students, 200 from each of the Biochemistry and Physics courses, participated in this study. Students were given an on-line test with 30 multiple-choice items. Faculty assigned a level of difficulty to each of the items ("Pass" - for simple, one-step problems, requiring recall or manipulation of information; "Credit" - for items requiring what a "Pass" question requires as well as the application of concepts to practical situations; "Distinction/High Distinction" - for questions requiring what a "Credit" question requires, as well as a higher level of analysis and critical thinking). Students answered the questions,

and assigned each question a level of difficulty (Pass, Credit, and Distinction/High Distinction). Results showed disagreement between student assessment of problem difficulty and faculty judgment of problem difficulty, as well as between student performance and student assessment of problem difficulty. Overall, the more difficult questions were associated with lower student and faculty agreement about difficulty levels, and longer response. For Biochemistry items, student and faculty agreement on problem difficulty was about the same for all questions. For Physics students, student and faculty agreement on difficulty was lower as student performance decreased (i.e., the agreement was 53% for "Pass" questions that were correctly answered by 71% of students, and the agreement decreased to 33% for "Distinction/High Distinction" questions that were correctly answered by only 47% of students). The association between student performance and student perception of problem difficulty was statistically significant only for Physics students. Physics students and faculty agreement of problem difficulty was significantly higher for those students who took Physics in high school. Overall, compared with the level of problem difficulty assigned by faculty, students who gave an incorrect answer to a question considered it to be more difficult, while students who correctly answered a question considered it to be less difficult. Furthermore, students who correctly identified a "Pass" question were more likely to answer it correctly. Students who correctly identified a "Distinction/High Distinction" question were more likely to answer it incorrectly.

Lingard et al.'s (2009) study is important because it draws attention to the low agreement on problem difficulty between students and their instructors. Furthermore, it highlights the complex relationships among a problem's specific domain of knowledge (e.g., Physics, Biochemistry), students' prior knowledge, student performance, student and faculty assessment of problem difficulty, and student and faculty agreement on problem difficulty. In other words, assessing the difficulty of a hard Physics problem by students without prior knowledge of Physics, or by their instructors, is a different task than assessing the difficulty of any Biochemistry problem by students or their instructors.

Gire and Rebello (2010) conducted two studies to investigate how factors, such as problem complexity, and familiarity with the type of problem, are correlated with student and teacher perceived problem difficulty. Participants in both studies were college students enrolled in introductory Physics courses and their instructors (professors, teaching assistants) at a Midwestern state university. Fifteen students, enrolled in a Calculus-based Physics course, and 14 instructors participated in the first study. Participants were asked to rate, on a scale from 1 to 10 (1-easiest, 10-most difficult), the level of difficulty of 16 problems on the topic of Work-Energy. Instructors rated the difficulty level of problems from the perspective of their students.

In the second study by Gire and Rebello (2010), students (N=21) were enrolled in an algebra-based Physics course. Students were asked to solve 10 Kinematics problems, as for example: The position of an abject moving along the *X*-axis is given by the polynomial function $x(t) = 3t^3 - 2t + 4$; find at least one time when

the velocity is zero. Subsequently, students rated the level of familiarity with the type of problem on a scale from1 to 5 (1-most familiar, 5-least familiar), and the level of difficulty for each problem on a scale from 1 to 5 (1-easiest, 5-most difficult). Instructors (N=16) of the students were asked to rate the level of problem difficulty, as perceived by their students. Student solutions to the Kinematics test were scored.

In the first study, with one exception, students rated five of the 16 problems as being easier than their instructors considered them to be. In the second study, students rated four of the 10 problems as being easier than their instructors' estimation of difficulty. Also, in the second study, student performance as well as student ratings of familiarity with problems correlated highly with student and instructor ratings of problem difficulty. High performance scores were associated with low ratings of problem difficulty; low ratings of problem difficulty were associated with high ratings of problem familiarity. In other words, students who correctly solved a problem were more likely to rate it as familiar and easy. In the first study, problem complexity correlated highly with instructor rating of problem difficulty, but not with student rating of problem difficulty. In the second study, problem complexity correlated highly with both student and instructor ratings of problem difficulty. Results concerning problem complexity suggest the necessity of reconsidering problem complexity scoring criteria, and that the problem domain (Work-Energy and Kinematics) and the place where the student is positioned in the mathematical curriculum course sequence (e.g.,

Calculus-based Physics, algebra-based Physics) may play important roles in student perception of problem difficulty.

Zhou (2009) used teacher assessment of relative difficulty of problems and the factors associated with those assessments, and compared them to the relative problem difficulty determined by student performance. The relative difficulty of a problem is given by its rank order in a set of problems. Participants were 10 teachers recruited from professional development workshops held at a university in the Southwest. All teachers had more than four years of teaching experience. Only two teachers held degrees in mathematics. Three of the 10 teachers had experience teaching only mathematics (Grades 7-9); the other seven teachers taught all elementary-school subjects, typical of elementary-certified teachers. Teachers were presented with 11 problems involving linear measurement (e.g., direct reading of a ruler, reading a ruler from starting positions other than zero, unit iteration, constant units, unit conversions, and unit partition to different fractional parts). They were asked to rank problems in order of difficulty from their students' perspectives. The test was also administered to elementary and middle school students (N=360) in several school districts by teachers participating in other professional development activities. Teacher ranking of problem difficulty and problem difficulty based on student performance were used to calculate the Spearman rank-correlations for each teacher. Only three teachers' rankings were significantly correlated with student performance. Two of those three teachers taught all subjects, and one teacher taught only middle-school mathematics. The latter teacher had the least number of years of teaching

experience. Overall, teachers were more likely to accurately estimate the relative ranking of problem difficulty for easier problems, i.e., problems most students could solve. Mathematical knowledge was not found to be associated with higher correlations between teacher ranking and student performance. One of the teachers with a high Spearman rank-correlation incorrectly solved one the problems. One of the two teachers with mathematics degrees overestimated easier problems, and the other underestimated the difficulty levels of problems. Neither the number of years of teaching experience nor the amount of professional development training was associated with higher correlations between teacher ranking and student performance.

The small number of teachers participating in this study (N=10) makes it difficult to generalize findings to other populations. The empirical ranking of problem difficulty was obtained using a sample of students from a population different form the population of students taught by the teachers participating in the study. This raises the issue of curriculum experiences that were more likely to be different for the two student populations, and therefore might invalidate the comparison intended by researchers. However, the study draws attention to the fact that the relationship between teachers' mathematical knowledge and pedagogical knowledge is a complex one, and more research is needed to investigate the factors associated with teacher understanding of their students' difficulties.

Nathan and Koedinger conducted studies (Koedinger & Nathan, 2004; Nathan & Koedinger, 2000) to compare teacher and researcher prediction of relative

problem difficulty (given by its rank order in a set of problems) with student performance. Participants were Grade 9 students from four U.S. urban high schools enrolled in Algebra I (N=229) and Geometry (N=18), experienced secondary mathematics teachers (N=67), and researchers in mathematics education (N=35). Teachers were recruited from professional development workshops held at a university in the southeastern part of the U.S. Researchers were recruited from an Internet discussion group. Student performance was assessed using an 8-item test of "early" algebra problems requiring quantitative reasoning. The comparison between teacher predictions and student performance focused on six types of problems determined by problem characteristics like the position of the unknown quantity ("result-unknown" or "start–unknown") and the format ("symbolic equation," "story problem," or "word-equation"). For example, one "start-unknown, word-equation" problem was:

> "Starting with some number, if I multiply it by 6 and then add 66, I get 81.90. What did I start with?" (Nathan & Koedinger, 2000, p. 170).

An example of a "result-unknown, symbolic equation" problem was:

> "Solve for *x*: (81.90 - 66) / 6 = *x*" (Nathan & Koedinger, 2000, p. 170).

A "start-unknown, story" problem was:

> When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour did Ted make? (Nathan & Koedinger, 2000, p. 170).

Based on student performance, “start-unknown” problems were more difficult than “result-unknown” problems, and “symbolic-equation” problems were more difficult than “word-equation” and “story” problems. Teachers and researchers were given 12 problems, two problems for each of the six types, and asked to rank them in the order of difficulty for their students, from easiest to most difficult. As predicted by teachers and researchers, “start-unknown” problems were more difficult for students than “result-unknown” problems. Contrary to teachers' and researchers’ expectations, who predicted that “story” problems and “word-equation” problems would be more difficult than “symbol-equation” problems, students found “symbolically-presented” problems most difficult. Nathan and Koedinger (2000) suggest that teachers' and researchers’ expectations of problem difficulty for their students reflect Algebra textbook instructional sequences in which “story” problems and “word-equation” problems are presented after “symbolic equation” problems.

Statistical significances reported by Koedinger & Nathan (2004) might not reflect the true relationships between problem characteristics and problem difficulty. The analysis of variance used by Koedinger & Nathan requires independence of observations, but this assumption was violated since participating students were grouped within the same classes and schools. Moreover, students were not asked about their perceptions of problem difficulty, and the comparison between student performance and teacher prediction of problem difficulty was made by a group of teachers different from those teaching the participating students.

Lorge and Kruglov (1953) hypothesized that information about the absolute difficulty of a subset of problems from a test, i.e., the percentage of students with correct answers for each of the problems in the subset, would lead to improved teacher assessment of the percentage of students with correct answers for all problems. Fourteen experienced mathematics teachers (about 10 years of teaching experience), pursuing advanced degrees in mathematics education participated in this study. Two 45-problem tests, on mathematics topics taken from Grade 8 and 9 mathematics textbooks, were used for this study. The empirical difficulty of each of the problems, given by the proportion of students in a specific population that correctly solved the problem, was determined by administering the test problems to samples of a minimum of 370 students in Grades 8 and 9. Teachers were informed about the percentage of students with correct answers for 10 of the problems on one of the tests, and they were asked to predict the percentage of students with correct answers for the rest of the questions in the tests. Although there was a slight improvement in teacher prediction of percentage of students with correct answers when information on student performance on 10 of the questions was given, teachers underestimated the percentage of students with correct answers. However, teachers were better at predicting the rank-order of problem difficulty in the two tests, as compared to the rank-order by student performance. The researchers recommended teacher access to information about student performance on subsets of problems from tests, in order to improve their ability to predict student scores with problems of those types.

2.7 Concluding Remarks

Solving equations was the purpose of algebra until the nineteenth century. Justifications for finding solutions to equations were mainly geometric in nature. With the exception of the New Math era, the Algebra I curriculum was dominated by solving equations using symbolic manipulations and formulas. After 1990, the static equation perspective began to be replaced by the dynamic functional perspective, and connections between graphic and symbolic representations. The function approach to modeling change emphasized the importance of the concept of rate of change, and brought to attention representations of real-world phenomena.

The review of research literature on student understanding linearity and linear functions revealed that students had difficulty with slope or rate of change, and making connections among representations, particularly the Cartesian connection. The geometric perspective of slope, i.e., slope as measure of the steepness of a line, generates more student difficulties than the analytical perspective of slope, that is, slope as rate of change. In the geometric perspective of slope, students have difficulty simultaneously attending to changes in y and x, and considering the change in y and change in x as oriented segments in the Cartesian system of coordinates. Students have difficulty connecting between various representations of functions, most notably, the Cartesian connection, i.e., connecting between the symbolic and the graphic representation of a linear function. Also, students have difficulty transferring knowledge between contexts (e.g., mathematical and real-life situations for linear functions, geometric and analytical/functional situations

for slope). Although with various degrees of intensity, research studies reveal that the same difficulties are experienced across various populations of students, regardless of age and previous mathematical experience. Moreover, researchers point out that the same difficulties also are experienced by pre-service and in-service mathematics teachers.

Research on teacher knowledge of student understanding of linearity and linear functions is scarce. The review of research literature on teacher understanding of student difficulties with linearity and linear functions shows that teachers' weak understanding of slope or rate of change and the Cartesian connection matches their weak ability to understand their students' thinking and attend to their students' difficulties with linearity and linear functions.

There is a paucity of research studies on student perception of problem difficulty as compared to student performance. Although the research on teacher assessment of problem difficulty as compared with student performance is somewhat richer, the various comparison types and research objectives make it difficult to use. The results of those studies indicate that the level of agreement between student and teacher perception of problem difficulty, and student performance varies with student and teacher cultural backgrounds, curricular experiences, type of learners, subject matter of the problem, and content and type of problem.

Chapter Summary. In this chapter, an historical perspective of algebra as a body of knowledge and algebra in the curriculum were presented, followed by a review of the research literature on student and teacher understanding of linearity

and linear functions, together with research on teacher understanding of student difficulties with aspects of linearity and linear functions, and research on student and teacher assessment of problem difficulty. In the next chapter, Chapter 3, data collected and methods of analyses are presented.

Chapter 3

DATA COLLECTED AND METHODS OF ANALYSES

3. 1 Introduction

The purpose of this study was to identify student difficulties with aspects of concepts of linearity and linear functions, and teachers' understanding of the nature of the difficulties students experience. To gain insight into students' and teachers' knowledge, instruments were either adapted from existing tools or developed particularly for this study, and then implemented to answer the following questions:

1. Does student performance on the Mini-Diagnostic Test (MDT) vary by:
    a. Grade level?
    b. Current mathematics course?
    c. Problem?
2. Does student ranking of problems by difficulty, after completing the MDT, correspond to their performance on the MDT?
3. Does teacher ranking of problems by difficulty, after completing the MDT, correspond to student performance on the MDT?
4. What is the nature of students' difficulties?
5. Do teachers' speculations about the nature of students' difficulties correspond to actual student difficulties as identified from student performance on the MDT and interviews?

The following are described in this chapter: the instruments used to collect data to answer the research questions, the pilot study designed to refine instruments and develop the Preliminary Coding Categories, the nature of the sample, the timeline for administration of the assessments, and the methods used to analyze the data for each research question. Assessment instruments are described in Section 3.2. The theoretical framework used for analysis is presented in Section 3.3. The pilot study, the Scoring Guide and Preliminary Coding Categories are presented in Section 3.4. Characteristics of participants, both students and their mathematics teachers, are described in Section 3.5. The timeline for recruiting participants, administering tests and interviews, scoring tests and coding interviews, and analyzing the data, is found in Section 3.6. Methods of analyses used to answer the research questions are addressed in Section 3.7.

3.2 Study Instruments

The following instruments were used to collect data for this study: the Mini-Diagnostic Test on Linearity and Linear Functions (MDT), the Ranking Questionnaire for Students (RQS) and the Ranking Questionnaire for Teachers (RQT), the Information Questionnaire for Students (IQS), the Information Questionnaire for Teachers (IQT), and the Interview questions for students and teachers.

The primary instrument for measuring students' understanding of linearity and linear functions is the Mini-Diagnostic Test (MDT). The MDT was developed by

Greenes et al. (2007b), a committee of mathematicians, mathematics educators, and school district mathematics leaders, after analyzing five years (2001-2005) of students' performance on linearity and linear functions items on the Grade 8 test of the Massachusetts Comprehensive Assessment System (MCAS).

The MDT, consisting of seven problems (the last problem with three sub-problems), uses three formats: multiple choice (problems 3, 5, and 6), short answer (problems 2, 4, and 7a, 7b, 7c) and extended response (problem 1). In the directions, participants are encouraged to show their solution methods along with their answers to the problems. MDT problem 1 gives the coordinates of a point and the equation of a line. The task is to determine whether the point is contained in the line, and to explain how the decision was made. MDT problem 2 requires construction of a table of five pairs of values (x_0, y_0) that satisfy a linear function presented in the form $y = mx + b$. MDT problem 3 provides a table of values for a linear function, and the task is to identify the corresponding graph from a group of four graphs. MDT problem 4 requires determination of the slope of a line from its graph. MDT problem 5 provides a table of values and five choices of linear functions presented symbolically. The task is to match the table of values with its corresponding function. MDT problem 6 involves figuring out the *y*-intercept of a line whose equation is given in the form $ax + by = c$. In MDT problem 7, a graph representing the movement of a car at three different speeds is presented along with three sub-problems or tasks. The tasks are to identify that part of the graph when the car is traveling the slowest, determine the slope of that part of the graph, and determine the speed of the car corresponding to that part of the graph. For

analysis purposes, each sub-problem of problem 7 was considered as a separate problem. The MDT requires about 30 minutes to complete. The MDT is reproduced in Appendix A.

After completing the MDT, all participants were asked to rank the MDT problems from most to least difficult. The Ranking Questionnaires were developed particularly for this study (see Appendix A). The Ranking Questionnaire for Students (RQS) asked them to order the MDT problems from most to least difficult, and comment on the nature of the difficulty for the most difficult problem. The Ranking Questionnaire for Teachers (RQT) asked them to order the MDT problems from most to least difficult as perceived by their students, and comment on the nature of the difficulty, as perceived by their students, for the two most difficult problems.

Ranking Questionnaires were designed to be administered along with the MDT. Necessary time for completion is 5 to 10 minutes.

Information Questionnaires were developed to identify several characteristics of the participants. The Information Questionnaire for Students (IQS) requests information about gender, grade level in school, current mathematics course, and current mathematics teacher. The Information Questionnaire for Teachers (IQT) requests highest academic degree, college major, current and previously taught mathematics courses, and number of years of teaching experience.

Information Questionnaires were administered along with the other assessment instruments, but before administration of the Mini-Diagnostic Test and the Ranking Questionnaires. The Information Questionnaires require 3–5 minutes for

completion. The total time required for completion of the Information Questionnaire, Mini-Diagnostic Test, and Ranking Questionnaire is 38-45 minutes.

To gain greater insight into their conceptual knowledge and nature of difficulties with MDT linearity concepts, interviews were developed for use with both student and teacher participants. These semi-structured, think-aloud, task based interviews required students to describe their solutions to all MDT problems to the interviewer. During students' descriptions of their solution processes, only comments of a prompting nature were used. Prompting comments did not interrupt the solution process but gave encouragement to "continue," or "tell more." Once a problem was solved, probing questions were used to elicit rationales about the choice of solution strategies or steps, or to satisfy the need for more information or clarification about the students' conceptual understanding of their difficulties. Probing questions included:

1. What is your meaning of slope?
2. How did you solve this problem? What did you do first? What did you do next?
3. Is there any other information you would have liked to have to solve this problem?
4. What was difficult about this problem? Why?
5. Why did you rank problem (*name problem that was ranked most difficult by the student*) as most difficult?

Semi-structured interviews were used with teachers, as well. All teachers interviewed for this study were asked to provide rationales for their selection of problems that would pose the greatest difficulty for their students, identify the knowledge students needed to solve those problems, and offer instructional strategies to overcome those difficulties. Teachers were asked questions like the following:

1. You chose problem (*name the problem the teacher chose*) as posing the greatest difficulty for students.
   a) What do students need to know in order to solve this problem?
   b) What might you do to help students overcome this difficulty?
2. Problem (*name the problem that caused difficulty*) has been difficult for many students.
   a) Why do you think they considered it to be difficult?
   b) What do students need to know in order to solve this problem?
   c) What might you do to help students overcome this difficulty?
3. How would you explain the meaning of slope to your students?

Probing questions were used to gain additional information or clarification for the answers to the questions above.

3.3 Theoretical Framework

The theoretical framework for analyzing student knowledge and difficulties with linearity and linear functions was inspired by Vergnaud's theory of conceptual fields (Vergnaud, 1994, 2009). The conceptual field is a set of

situations, schemes for dealing with situations, sets of concepts and theorems necessary to analyze the operations of thinking, and sets of representations, formulations and symbolizations (Vergnaud, 1994).

The Linear Conceptual Field (LCF) consists of:

1. Situations that can be modeled mathematically using linear relationships or linear functions (e.g., rectilinear movement with constant speed as in MDT problem 7). Some situations have real-life contexts (e.g., MDT problems 5 and 7), while others have mathematical contexts (e.g., MDT problems 1, 2, 3, 4, and 6).

2. Schemes of actions (e.g., identifying the coordinates of a point situated on a line graphed in a system of coordinates in MDT problem 4), or theorems in action (e.g., the Cartesian connection - a point (x_0, y_0) is on the graph of the line *l* if and only if its coordinates satisfy the equation of *l*, $y = mx + b$) needed to solve problems (e.g., MDT problems 1 and 2). Some schemes of action, used to correctly solve a problem, are in fact algorithms for solving the problem. For example, in MDT problem 1, to check if a given point (x_0, y_0) satisfies the equation of the line, the solver needs to substitute the coordinates of the given point into the equation of the line, $y = mx + b$, and check whether the statement $y_0 = mx_0 + b$ is true.

3. Concepts of linearity and linear function (e.g., concept of slope in geometric-analytical context in MDT problems 4 and7b, concept of speed in MDT problems 7a and 7c), and ways of thinking (e.g., quantitative reasoning).

4. Representations of linear functions (e.g., informal, natural language, or formal mathematical language; symbols, tables, graphs, diagrams). The MDT problems contain various representations of linear functions (e.g., symbolic rule in MDT 1, table of values and graphic representation in MDT problem 3).

3.4 Pilot Study

The primary purpose of the pilot study was to gain insight into the nature of students' difficulties with MDT problems, and teachers' understanding of their students' difficulties, in order to adapt the Scoring Guide (SG), and develop Preliminary Coding Categories (PCC) for the RQS, RQT and Interviews.

Participants in the pilot study were 213 Algebra I students in Grades 8-10 and their four mathematics teachers. Ninety-nine students were in Grade 8, one-hundred and six students were in Grade 9, and only eight students were in Grade 10. All Grade 8 students were enrolled in the same middle school. All Grade 9 and 10 students were enrolled in the same high school. Both schools were in the same school district that is primarily middle class (less than 25% of students eligible for free or reduced-price lunch). The MDT and Ranking Questionnaires (RQ) were administered to all participants. Four teachers and 77 students agreed to be interviewed. All four teachers were interviewed, together with 10 students (five students in Grade 8, four students in Grade 9, and one student in Grade 10). Interviewed students were selected by the researcher, such that half of them had MDT scores above the median, and half had MDT scores below the median.

Student MDTs were scored using the Scoring Guide. The Scoring Guide is described in Section 3.4.1. The Preliminary Coding Categories for Ranking Questionnaires (RQ) are presented in Section 3.4.2. The Preliminary Coding Categories for Interviews are presented in Section 3.4.3.

3.4.1 Scoring Guide (SG) for the MDT. The Scoring Guide (SG) (see Appendix A) is a modification of the Preliminary Scoring Guide (PSG) developed by Greenes et al. (2007b) in Massachusetts. Initially each of the seven problems (1, 2, 3, 4, 5, 6, and 7) of the MDT was scored 20 points. The PSG modifications involved reconsidering the number of problems for statistical purposes, and scaling to facilitate interpretation of the dichotomous scores. The analysis of the correlations between the MDT problems led to considering nine MDT problems (1, 2, 3, 4, 5, 6, 7a, 7b, 7c), rather than seven. Six of the MDT problems (3, 4, 5, 6, 7a, 7b) have dichotomous scores. The initial dichotomous scores 0 or 20 were changed to 0 or 1. One of the advantages of using dichotomous scores (0 and 1) is the interpretation of the means for MDT problems 3, 4, 5, 6, 7a, 7b as percentages of participating students with correct answers. Another advantage is the quick identification of outliers signaled by the presence of means greater than .9. It was expected that students in advanced mathematics classes, like Algebra II, had mastered the MDT aspects of linearity and linear functions. Thus, the MDT would be easy for them, and consequently their MDT scores would be high (> .9), creating the ceiling effect (McBee, 2010).

Also, the Preliminary Scoring Guide was modified for use in this study to represent participants' types of responses to Problems 1 and 4. The adaptation was minor. A description of the adaptation follows.

For MDT problem 1:

a. The modification consisted of including the type of answer different from "Yes" or "No" for Problem 1a; and
b. Reformulating the descriptions of types of answers for Problems 1b.

For MDT problem 4:

a. Separating the types of answers "Irrelevant information," "No answer," and "I don't know" into two disjoint categories, "Irrelevant information or no answer," and "I don't know;"
b. Eliminating the type of answer "Computing $\frac{\Delta y}{\Delta x}$ with incorrect coordinates;" and
c. Reformulating the descriptions of types of answers.

The modified Scoring Guide, described below, will be referred to, from now on, as the Scoring Guide (SG). Table 1 presents score possibilities for the nine MDT problems.

Table 1

MDT Scores by Problem

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Problem 7		
						a	b	c
0, .5 or 1	0, .2, .4, .6, .8 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0, .5, or 1

As can be seen in Table 1, MDT problems 3, 4, 5, 6, 7a and 7b are scored 0 for an incorrect response or no response, and 1 for a correct response. MDT problem 2 is scored 0, .2, .4, .6, .8 or 1, based on the number of correctly completed rows (0 to 5) in the table of values. MDT problem 1 is separated into two parts, part 1a scored 1 if the answer is "Yes," and 0 otherwise, and part 1b is scored 1 for a correct explanation, and 0 otherwise. The total score for MDT problem 1 is obtained by adding the scores for parts 1a and 1b and dividing the sum by 2. MDT problem 7c is separated into two parts, 7c1 and 7c2, one part for the numeric value and the other for the unit, with 1 point for each correct answer and 0 otherwise. The total score for MDT problem 7c is obtained by adding the two scores for parts 7c1 and 7c2 and dividing the sum by 2 to produce an average score for that problem. The high correlations between the pairs of scores for the parts 1a and 1b (.93), and parts 7c1 and 7c2 (.80), respectively, explain the need for collapsing the two-parts scores into single scores for MDT problems 1 and 7c, respectively. Possible scores for MDT problems 1 and 7c are 0, .5, or 1. For each of the MDT problems, 1 through 6, 7a, 7b, and 7c, scores range from 0

through 1. The MDT total score is obtained by adding scores for all problems 1, 2, 3, 4, 5, 6, 7a, 7b, 7c, and ranges from 0 through 9.

Types of responses for MDT problems 1, 2, 4, 7a, 7b, and 7c with short answers or extended responses are presented in Tables 2-7. Table 2 presents types of answers for MDT problem 1 that requires determination if a given point (*x*, *y*) is situated on a line.

Table 2

MDT Problem 1 Types of Responses

PART	1. *Correct answer "Yes"*
(a)	2. Answer "No"
	3. No answer or an answer different from "Yes" or "No"
PART	1. *Correct answer using the Cartesian connection*
(b)	Identifying $(2,-8)$ as (x_0, y_0) and substituting in $y = 3x - 14$ *;* and Checking if the statement $-8 = 3(2) - 14$ (or equivalent statement) is true.
	2. Incorrect or incomplete answer while attempting to use the Cartesian connection
	3. *Correct answer using a graph*
	Graphing the line $y = 3x - 14$ (using a quick graph or a table of values); Plotting the point $(2,-8)$; and Inspecting to see if the point is on the line.
	4. Incomplete or incorrect answer from use of a graph
	5. Incomplete or incorrect responses different from those above from Part (b), #1-4
	6. No answer or wrote "I guessed"

As can be seen from Table 2, the type of response for MDT problem 1a is separated into three non-overlapping categories: "Yes," "No," and no answer or an answer different from "Yes" or "No." Based on analyses of the performances of the participants in this present study two problem solving strategies used by participants who solved MDT problem 1b correctly were identified. These strategies are those of the Cartesian connection and the use of a graph, and they imply two different algorithms for solving MDT problem 1. The Cartesian connection strategy implies the identification of the Cartesian coordinates of the point $(2,-8)$ as (x_0, y_0) in the equation of the line $y = 3x - 14$, and checking whether the equality $-8 = 3(2) - 14$ is a true statement. The strategy of using a graph implies that the line $y = 3x - 14$ is graphed, and the solver determines that the point $(2,-8)$ lies on the graphed line. Scoring MDT problem 1b requires determination of the correctness of the answer and the solution strategy. Six non-overlapping coding categories were identified during the coding process: correct answer using the Cartesian Connection, incorrect answer using the Cartesian Connection, correct answer using a graph, incorrect answer using a graph, and no answer or "I guessed" answer.

Table 3 presents types of responses for MDT problem 2 that requires participants to construct five rows of a table of values, given the linear function $y = -2x + 3$.

Table 3

MDT Problem 2 Types of Responses

1. *5 rows correctly completed*
2. 4 rows correctly completed
3. 3 rows correctly completed
4. 2 rows correctly completed
5. 1 row correctly completed
6. No row correctly completed, but attempted to answer
7. No answer

The types of responses for MDT problem 2 presented in Table 3 were determined by the number of correctly completed rows in the table of values. Response types 6 and 7 differentiated between participants who attempted to answer with no success from those who did not attempt. This differentiation was necessary to separate out those who do not answer correctly due to computational errors or lack of understanding of how to complete a table of values for a function from those who do not try to complete the table.

Table 4 presents types of responses for MDT problem 4 that requires identification of the slope of a line graphed in a homogeneous system of coordinates.

Table 4

MDT Problem 4 Types of Responses

1. *Correct answer "3"*
2. Computational errors in calculation of slope, but correct formula
3. Reversed ratio
4. Incorrect answer different from answers to 2 and 3
5. Irrelevant information
6. No answer or answer, "I don't know."

Types of incorrect answers for MDT problem 4 as shown in Table 4 are based on most common errors observed in previous studies conducted by Greenes et al. (2007a, 2007b) (e.g., errors in calculation of slope due to incorrect computations with signed numbers, or presentation of slope in "upside down" or reversed order, that is, change in x over change in y). The Type 5 code ("Irrelevant information") is applied when, for example, qualitative comments (e.g., "increasing," or "positive") are made without any attempt to slope calculation. Type 6 code is used when no answer is given, or a statement is made to the effect that the solver does not know what to do.

Table 5 presents types of responses for MDT problem 7a. That problem requires identification of the part of the graph when a car moves the slowest.

Table 5

MDT Problem 7a Types of Responses

1. Answer "R"
2. Answer "S"
3. *Correct answer "T"*
4. Answer different "R", "S", or "T "
5. No answer

As can be seen from Table 5, types of responses for MDT problem 7a are grouped into five disjoint categories ("R," "S," "T," an answer different from those mentioned before, and no answer).

Table 6 presents the types of responses for MDT problem 7b, that requires determination of the slope of a segment of a line graphed in a non-homogeneous system of coordinates.

Table 6

MDT Problem 7b Types of Responses

1. *Correct answer "40"*
2. Incorrect answer
3. No answer or irrelevant information

As can be seen in Table 6, there are three types of responses. All incorrect answers resulting from attempts to calculate the slope are grouped together under

"Incorrect answer." Other types of answers that present irrelevant information or show no work are grouped together as a Type 3 response.

Table 7 presents the types of responses for MDT problem 7c that requires determination of the speed of the car.

Table 7

MDT Problem 7c Types of Responses

1. *Correct answer "40mph"*
2. Correct numeric answer, but incorrect or no unit of measurement given
3. Incorrect answer
4. No answer or irrelevant information

As can be seen in Table 7, when the numeric part of the answer is given and either the measurement unit is incorrect or not given, the response is considered to be of Type 2. When an attempt to calculate the speed is evident but the answer is incorrect, the response is coded as Type 3. As with other problems, Type 4 comprises responses when no indication of the correct computation is evident or when no response is given.

3.4.2 Preliminary Coding Categories for Ranking Questionnaires. For the Ranking Questionnaires described in Section 3.2 students and teachers commented on the nature of student difficulties with MDT problems. Students' descriptions were short, usually only one sentence, and fairly repetitive. Teachers' descriptions were also short, mostly one or two sentences in length. The

researcher read each comment and identified the major idea. For comparison purposes, both students' and teachers' written comments were coded using the same Preliminary Coding Categories (PCC) for Ranking Questionnaires (RQ), presented in Table 8.

Table 8

Preliminary Coding Categories for RQ Written Responses

Category	Description	Frequency (Percentage of Total N=213)
H (Hard Problem)	A statement about the difficulty of a problem or concept.	16 (7.5%)
M (Mathematical Difficulty)	The nature of difficulty refers to lack of understanding of mathematical concepts, their representations, or algorithms of solving a problem.	19 (8.9%)
R (Not Remembering)	A statement about forgetting or not remembering the mathematical concepts or how to solve the problem.	5 (2.3%)
U (Not Understanding)	A general statement about not understanding a mathematical concept, how to solve a problem, without offering any further details about the nature of the difficulty.	97 (45.5%)
W **(**Whole Domain**)**	A statement about the difficulty of a whole domain like algebra or mathematics, in general, with no specifics.	2 (0.9%)
O **(**Other Difficulties**)**	Other difficulties, different from the ones in the categories mentioned above.	4 (1.9%)
No Comment		70 (32.9%)

As can be seen in Table 8, there were six preliminary coding categories: *H, M, R, U, W* and *O***.** Category *H* ("Hard Problem") was used with students' comments about general problem difficulty using expressions like, "It's a hard problem."

Category *M* ("Mathematical difficulty") was used when specific difficulties with mathematical concepts were identified. Category *R* ("Not Remembering") was used when students made comments about forgetting ("I don't remember"). Category *U* ("Not Understanding") was used when students commented "I don't understand it." Category *W* ("Whole Domain") was used with comments about whole domains of mathematics, like algebra, or mathematics (e.g., "I'm not good at Algebra," or "Math is hard for me"). Category *O* ("Other Difficulties") was used for difficulties that did not fit into the other categories, like references to difficulty with vocabulary or lack of sufficient time to solve the problem. Of note, about half of the students (45%) made comments of "Not Understanding," about one third of students (32.9%) did not make any comments, and frequencies of responses in categories *W* and *O* were small, 2 and 4, respectively.

3.4.3 Preliminary Coding Categories for Interviews. The most difficult problems for pilot study students were MDT problems 7b and 4, both requiring determination of slopes of lines, and MDT problem 1, considering whether a point identified with its coordinates is on a line, given the equation of the line. Only 28% of pilot study students solved MDT problem 7b correctly, 33% solved MDT problem 4 correctly, and 53.5 % solved MDT problem 1 correctly. Consequently, the interviews focused on student knowledge and difficulties with the Cartesian connection and slope.

Pilot study data, along with results of research studies (Lobato & Siebert, 2002; Moschkovich 1992, Moschkovich et al., 1993; Schoenfeld et al., 1993;

Stump 1997, 2001a, 2001b; Zaslavsky et al., 2002), were used to develop Preliminary Coding Categories for analyzing data from Interviews. The Preliminary Coding Categories were developed by the researcher using open coding at the level of paragraph (Strauss & Corbin, 1998). Answers to the researcher's questions were considered to be the paragraphs. Responses that did not address the question were excluded from analysis. Tables 9 and 10 present the Preliminary Coding Categories referring to conceptual knowledge of Cartesian connection and slope, respectively.

Table 9

Preliminary Coding Categories for Conceptual Knowledge of Cartesian Connection

CKC Categories	Description
CKC-D Cartesian Connection from Point to Line	A point (x_0, y_0) is on the graph of the line *l* if its coordinates satisfy the equation of *l*, $y = mx + b$.

Of note, in Table 9 the Cartesian connection is stated as a mathematical theorem. However, when solving MDT problem 1, none of the interviewed participants referred to the mathematical theorem, but rather, to the algorithm for solving the problem. Three of the interviewed students missed the Cartesian connection and failed to solve MDT problem 1.

Table 10 presents the Preliminary Coding Categories for conceptual knowledge of slope.

Table 10

Preliminary Coding Categories for Conceptual Knowledge of Slope

CKS Categories	Description
CKS-P Physical Context	Slope describes the steepness of physical objects such as mountain roads, ski slopes, and wheelchair ramps. A statement about an entire class of objects sharing the property of having a slope or being steep, or an instantiation/particular example is given.
CKS-G Geometric-Analytical Context	Given two points (x_1, y_1) and (x_2, y_2), the slope of the line connecting the two points is $m = \frac{y_2 - y_1}{x_2 - x_1}$ (or "rise over run"). Homogeneous Systems of Coordinates: Slope is a geometric property of the line $y = mx + b$, as well as an analytical property of the function $y = mx + b$. Non-homogeneous Systems of Coordinates: Slope is an analytical property of the function $y = mx + b$, and not a geometric property of the line graphed.
CKS-A Parametric Context	Slope is the parameter *m* in the function $y = mx + b$.
CKS-T Trigonometric Context	Slope is the tangent of the angle of inclination θ, $m = \tan\theta$.
CKS-C Covariational Context	Slope represents the constant rate of change $m = \frac{\Delta y}{\Delta x}$ when linear functions in the form $y = mx + b$ are understood from a covariational point of view: two quantities varying simultaneously with a constant rate of change.

As can be seen in Table 10, there are five contexts (physical, geometric-analytical, parametric, trigonometric, and covariational) for slope. The interviewed participants from the pilot study referred to slope mainly in physical and geometric-analytical contexts. There was only one reference to slope in each

of the other contexts. One student mentioned slope as "m in $y = mx + b$," one teacher referred to slope as "tangent of an angle," and another teacher referred to slope as "rate of change."

Tables 11 and 12 show the Preliminary Coding Categories representing student difficulties with the Cartesian connection and slope in MDT problems 1, 4 and 7b.

Table 11 describes student difficulties with the Cartesian connection in MDT problem 1.

Table 11

Preliminary Coding Categories for Student Difficulties with the Cartesian Connection

Categories	Subcategories	Description of Difficulty
SDC-D *Student Difficulties with Cartesian Connection from Point to Line* A point (x_0, y_0) is on the graph of the line *l* if its coordinates satisfy the equation of *l*, $y = mx + b$, i.e., if the statement $y_0 = mx_0 + b$ is a true statement.	*SDC-P* Identifying the Point	Identifying $(2,-8)$ as (x_0, y_0) and replacing variables with their values in $y = 3x - 14$.
	SDC-T Checking if True statement	Checking if the statement $-8 = 3(2) - 14$ (or other equivalent statement) is true.

Table 11 shows conceptual knowledge of the Cartesian Connection from point to line. Initially stated as a mathematical theorem (a point (x_0, y_0) is on the graph of the line *l* if its coordinates satisfy the equation of *l*, $y = mx + b$), the Cartesian connection was replaced by a "theorem in action" (Vergnaud, 2009) , i.e., a point

(x_0, y_0) is on the graph of the line *l* if the statement $y_0 = mx_0 + b$ is a true statement. The Cartesian connection from point to line was observed in students' answers and teachers' comments only as a "scheme-in-action," a two-step algorithm for solving MDT problem 1 (identifying $(2,-8)$ as (x_0, y_0), replacing the variables *x*, *y* with their corresponding values x_0, y_0 in $y = 3x - 14$, and checking whether the statement $-8 = 3(2) - 14$ is true). Students' difficulties occurred at one or more of the steps of the algorithm: substitution (one student), or verification that the statement is true (one student). Three of the interviewed teachers did not identify MDT problem 1 as being difficult for their students. The fourth teacher suggested that MDT problem 1 might be difficult for students because of the need to explain their answers.

Table 12 shows the Preliminary Coding Categories for student difficulties with slope in the contexts of MDT problems 4 and 7b. Since the only students having difficulty with slope were students with geometric perspectives of slope, i.e., slope is a geometric attribute of a line, Table 12 details only those geometric difficulties. Students attempted, unsuccessfully, to calculate the slope of the oblique line using "slope triangles," i.e., triangles with two vertices being points on the line, (x_1, y_1) and (x_2, y_2), and the third vertex being either (x_1, y_2) or (x_2, y_1). When using a "slope triangle" to measure the rise and the run, the rise and the run are oriented segments with positive or negative magnitudes (Schoenfeld et al., 1993). Coordinating the unit of measurement with the length of the line segment to be measured implies having a clearly established starting

point, or origin of measurement, and then imagining or concretely laying down as many non-overlapping, adjacent line segments of one unit length as necessary to cover the entire line segment to be measured ("exact" measurement, when the line segment to be measured has a magnitude expressed by an integer). The transport of the unit of measurement is rigid during the measurement process, i.e., the unit of measurement is invariant. As mentioned before, in homogenous systems of coordinates the unit of measurement is the same for both the *X*-axis and the *Y*-axis, while in non-homogeneous systems of coordinates the unit of measurement for the *X*-axis and the *Y*-axis are different. All interviewed students who encountered difficulties measuring rise or run from "slope triangles" failed to coordinate the origin, unit, or sense of measurement.

All other students, with analytical perspectives of slope, i.e., slope as an attribute of the linear function represented graphically by a line, did not encounter difficulties. Those students calculated the slope of the line using the following algorithm: 1) identify the Cartesian coordinates (x_1, y_1) and (x_2, y_2) of two points on the line, and 2) calculate $m = \frac{y_2 - y_1}{x_2 - x_1}$. The importance of holding an analytical perspective of slope is crucial in MDT problem 7b where non-homogeneous systems of coordinates are used (Zaslavsky et al., 2002).

Table 12

Preliminary Coding Categories for Student Difficulties with Slope

Categories	Subcategories	Description
SDS-R Making and Calculating a Ratio Making the rise (change in *y*), run (change in *x*), their ratio (slope), and calculating the ratio "rise over run"	*SDS-MH Measurement in a Cartesian Homogeneous System of Coordinates*	Identifying the rise and the run, and their magnitudes by measurement in a "slope triangle" (a triangle with two vertices being points on the line, (x_1, y_1) and (x_2, y_2), and the third vertex being either (x_1, y_2) or (x_2, y_1)). Measurement refers to coordinating the origin point and the endpoint of measurement, the unit of measurement, and the sense of the measurement.
	SDS-MN Measurement in a Cartesian Non-Homogeneous System of Coordinates	Identifying the unit of measurement by taking in consideration the geometric representation of the line and counting the units on the *Y*-axis, instead of using values of the function represented by the line.
	SDS-CR Calculating a Ratio	Calculating the ratio "rise over run."

"Making a Ratio" in Table 12 refers to the process of conceiving a function of an ordered pair of magnitudes, while calculating a ratio refers to the numerical operation of evaluating the value of the function for a particular pair of magnitudes (Freudenthal, 1983). All three interviewed students who had difficulty with slope used the geometric context for slope ("rise over run"). The first interviewed student with slope difficulties, could identify the rise but considered the run to be zero, and could not calculate the slope as "rise over run" because she knew she could not divide by zero (MDT problem 4). The second student incorrectly measured the run (1.5 units), while the third student correctly

calculated the slope in MDT problem 4 using "rise over run" ($m = 3$), but incorrectly in MDT problem 7b ($m = \frac{4}{3}$, instead of $m = 40$). None of the teachers identified MDT problems 4 as being difficult for students. Three teachers referred to MDT problem 7 (7a, 7b, and 7c) as being difficult because of the presence of the graph with three parts, making it difficult to interpret. One teacher pointed out that the numbers are "too big," making calculations difficult.

Pilot-study students had difficulty determining the rise (change in *y*), run (change in *x*), and the ratio (slope). Their difficulties were observed as "schemes in action" (Vergnaud, 2009), that is, as difficulties with measurement in a "slope triangle" (identifying the origin, the unit of measurement in homogeneous and non-homogeneous systems of coordinates, and the sense, positive or negative, of the axes of coordinates). None of the pilot-study teachers identified the nature of their students' difficulties with slope in geometric context.

3.5 Participants

Participants included 1,565 students in Grades 8, 9, and 10, and their 27 mathematics teachers from two school districts in a state in the Southwest U.S. Of note, data analyses were conducted with 1561 students and 26 teachers. Several characteristics of the participants and their school districts are presented in this section. District level data were collected from the districts' official websites at the time of data collection. Data at the school and individual level

were collected from the Information Questionnaires completed by students and teachers.

Demographics of the two districts are shown in Table 13.

Table 13

Demographics by School District

	Hispanic	White	African-American	Native-American	Asian	Students Eligible for Free or Reduced-Price Lunch
District 1	29%	57%	6%	1%	7%	29%
District 2	18%	71%	5%	1%	4%	2%

As can be seen in Table 13, the majority of the population in both districts is White, 57% and 71%, respectively. There is a large difference in socio-economic status. The percentage of students eligible for free or reduced lunch is 29% in District 1 and only 2% in District 2.

3.5.1 Student Demographics. Numbers of students participating in this study by district and by school (middle school, Grades 6 – 8, and high school, Grades 9 – 12) are presented in Table 14. Note that middle schools are included because Algebra I is often taught in Grade 8.

Table 14

Student Distribution by District and School

District	School	Number of Students
District 1	Middle School A	259
	High School B	263
		Total 522
District 2	Middle School C	494
	High School D	549
		Total 1043

From Table 14 it can be seen that there were twice as many participating students from District 2 (N=1043) than from District 1 (N=512), that is, almost two thirds of all students in this study (66.6%) were from District 2. About half of the students from each district (259 from District 1, and 494 from District 2) were 8th graders in middle school, and half were 9th and 10th graders in high school (263 from District 1, and 549 from District 2). Data collected about gender revealed an almost equal split: 53.3% girls, and 46.7% boys.

Student distribution by grade and current mathematics course is presented in Table 15.

Table 15

Student Distribution by Grade and Current Mathematics Course

Grade	8	9	10	
Course				Total per Course
Pre-Algebra	457 [0]			457
Algebra I	286[1]	281[0]	8 [0]	575
Geometry	14[1]	135[1]	306[0]	455
Algebra II			78[1]	78
Total per Grade	757	416	392	Grand Total 1565

Note. [0] identifies Group 0 students; [1] identifies Group 1 students.

As can be seen from Table 15, there are two main groups of students:

i) Group 0 students, representing 67% of the study participants, are enrolled in mathematics courses at grade level or below grade level, including Grade 8 students enrolled in Pre-Algebra (N=457), Grade 9 students enrolled in Algebra I (N=281), and Grade 10 students enrolled in Algebra I (N=8) or Geometry (N=306) .

ii) Group 1 students, representing 33% of the study participants, are enrolled in more advanced mathematics courses than their grade level indicates, including Grade 8 students enrolled in Algebra I (N=286) or Geometry (N=14), Grade 9 students enrolled in Geometry (N=135), and Grade 10 students enrolled in Algebra II (N=78).

Number of participating students by mathematics course, school, and teacher are presented in Table 16.

Table 16

Participating Students by Current Mathematics Course, School, and Teacher

Current Mathematics Course	School	Teacher	Number of Students	Total
Pre-Algebra	Middle School C	Teacher 1	118	
		Teacher 2	72	
		Teacher 3	73	
		Teacher 4	98	
	Middle School A	Teacher 26	96	
				457
Algebra I	Middle School C	Teacher 5	47	
		Teacher 6	58	
		Teacher 7	28	
	High School D	Teacher 8	60	
		Teacher 9	20	
		Teacher 10	19	
		Teacher 11	28	
		Teacher 12	7	
		Teacher 15	18	
	High School B	Teacher 21	65	
		Teacher 22	72	
	Middle School A	Teacher 25	126	
		Teacher 27	27	
				575
Geometry	High School D	Teacher 8	20	
		Teacher 11	39	
		Teacher 12	18	
		Teacher 13	3	
		Teacher 14	60	
		Teacher 15	37	
		Teacher 16	23	
		Teacher 17	74	
		Teacher 18	45	
	High School B	Teacher 23	38	
		Teacher 24	84	
	Middle School A	Teacher 27	14	
				455
Algebra II	High School D	Teacher 13	1	
		Teacher 19	53	
		Teacher 20	24	
				78
				Grand Total 1565

As can be seen in Table 16, seven District 1 (Middle School A and High School B) teachers (Teachers 21-27) and 20 District 2 (Middle School C and High School D) teachers (Teachers 1-20) participated in this study. Grade 8 student participants were enrolled in courses ranging from Pre-Algebra (Teachers 1-4, 26) to Algebra I (Teachers 5-7, 25, 27) and Geometry (Teacher 27). Some teachers, like Teachers 8, 11, 12, 15, 27, taught both Algebra I and Geometry. Since Teacher 13 had few students who participated in this study, three in Geometry and one in Algebra II, data on those participants were eliminated from all statistical analyses. Consequently, data analyses were conducted with 1561 students and 26 teachers.

In all participating schools, students were placed in mathematics courses based on their prior performance in mathematics as reflected by course grades and teacher recommendations, not by age. Therefore, students of various ages and abilities could be in the same class with the same teacher.

3.5.2 Teacher Demographics. Table 17 shows number of years of teaching experience and highest college degree attained by teachers participating in this study.

Table 17

Participating Teachers' Experience and Degree

Participating Teachers		Experience < 4 years	Experience 4 to 6 years	Experience 7 to 9 years	Experience > 9 years	Bachelor	Master
District 1	Middle School (N=3)	0%	0%	33%	67%	33%	67%
	High School (N=4)	0%	50%	25%	25%	25%	75%
District 2	Middle School (N=7)	29%	14%	14%	43%	71%	29%
	High School (N=13)	46%	8%	8%	38%	54%	46%
Overall (N=27)		33%	19%	11%	37%	56%	44%

From Table 17 it can be seen that, with the exception of District 1 Middle School, the majority of teachers participating in this study had less than nine years of teaching experience (63%). Comparing districts, District 2 had fewer teachers with Master degrees than District 1, both at the middle school and high school levels, and more inexperienced teachers, with 29% of middle school teachers and 46% of high school teachers having less than four years of teaching experience.

Only one-third of the middle school teachers and three fourths of the high school teachers participating in this study had degrees in Mathematics or Mathematics Education. Advanced mathematics courses (e.g., Algebra I, Geometry in middle schools; Algebra II, Precalculus, Calculus in high schools) were taught primarily by more experienced teachers with higher academic degrees (e.g., Master's in Mathematics Education).

3.6 Timeline for Instrument Development, Recruitment of Participants, Data Collection and Analyses

The pilot study, development of the instruments and interview protocols, the recruitment of participants, the data collection and the data analyses were performed according to the schedule shown in Table 18.

Participant recruitment included meetings with district leaders, school principals, department chairs, teachers, students, and their parents. There were multiple site visits, discussions, phone calls, and e-mails. The recruitment materials and the IRB approvals can be found in Appendix C.

Table 18

Timeline for Instrument Development, Recruitment of Participants, Data Collection and Analyses

Time Period	Action
June 2008- November 2008	Pilot study conducted and data analyzed Development of Preliminary Coding Categories
October 2008–December 2008	Adaptation of the Scoring Guide for the Mini-Diagnostic Test, Development of the instruments RQS, RQT, IQS, IQT, Interview Questions
December 2008 –February 2009	Recruitment of participants
January 2009 –February 2009	Administration of the MDT, RQS, RQT, IQS, IQT, Scoring MDTs
February 2009–March 2009	Administration of interviews
April 2009–January 2010	Data Analyses performed Refinement of the Preliminary Coding Categories, development of the Final Coding Categories Inter-Rater Reliability conducted

As can be seen in Table 18, the conduct of this study began in June 2008 and concluded in January 2010.

The teachers who agreed to participate in this study invited their students to participate, as well. Student participation was meritorious; overall, more than 70% of those invited agreed to participate. The Mini-Diagnostic Test (MDT) on linearity and linear functions was distributed to all Grades 8, 9, and 10 students and their mathematics teachers participating in this study, during January and February 2009. The MDT was administered to participating students by their teachers who were instructed by the researcher about how to administer the test, and obtained their verbal agreement to observe test-like conditions. The MDT was administered to participating students during regular class time. Participating teachers solved the MDT problems at their convenience, and all were done within six weeks of student completion of the MDT. The Information Questionnaires and Ranking Questionnaires were administered at the same time as the MDT. The Mini-Diagnostic Tests were scored by the researcher using the Scoring Guide. The written comments from Ranking Questionnaires were coded using the Final Coding Categories (see Appendix B), and inter-rater agreement analyses of the coding were performed.

Interviews with participants were conducted within six weeks from the time of the administration of the MDT. Twenty teachers from District 2, seven middle school teachers and 13 high school teachers, agreed to be interviewed. Based on their responses to the MDT, two students from each of the 20 teachers were selected by the researcher and invited for interviews. In each pair of students, one

student had the MDT total score above the group median, and one student had the MDT total score below the group median. Interviews were conducted by the researcher during school time, were audio-taped, and were of 20 minutes duration. All interviews were transcribed after the data were reduced and coded by the researcher and another rater, using the Final Coding Categories developed for this study (see Appendix B). Inter-rater agreement analyses of the coding were performed.

3.7 Methods of Analyses

In this section, the methods of analyses used to answer Research Questions 1-5 are presented, by research question.

Research Question 1:

Does student performance on the Mini-Diagnostic Test (MDT) vary by:

a. Grade level?
b. Current mathematics course?
c. Problem?

Data were analyzed using descriptive statistics, cluster analysis and factor analysis. To analyze how student performance on MDT varies by grade and current mathematics course, descriptive statistics (calculations of means and standard deviations) were used. The analysis of MDT scores using descriptive statistics led to the consideration of three possible natural clusters: by grade level, current mathematics course, and performance level. A comparison of the quality of clusters, using the largest overall average silhouette (Rousseeuw, 1987) as a

criterion for determining the best cluster, was performed. To analyze how student performance on the MDT varies by problem, an Exploratory Factor Analysis (EFA), R-type, using the Principal Component Analysis method (Hair, Anderson, Tatham, & Black, 1998) was employed. The R-type EFA "analyzes relationships among variables to identify groups of variables forming latent dimensions (*factors*)" (Hair at al., 1998, p. 90). An EFA using the Principal Component Analysis method is a "factor model in which the factors are based on the total variance." It "implies that all variance is *common* or shared" (Hair et al., 1988, p. 88). The Principal Component Analysis method was preferred because the main goal of EFA was to reduce the original set of MDT problems to a new set, with fewer number of factors, that would be representative in subsequent analyses for all MDT problems from the old set.

Research Questions 2 and 3:

2. Does student ranking of problems by difficulty, after completing the MDT, correspond to their performance on the MDT?
3. Does teacher ranking of problems by difficulty, after completing the MDT, correspond to student performance on the MDT?

For Research Questions 2 and 3, to assess the association between the ranking of MDT problems by their mean scores and the ranking of MDT problems by perceived difficulty, the coefficient of correlation Kendall-tau was used. Kendall-tau is a non-parametric correlation coefficient that "may be defined as the

probability of concordance minus the probability of discordance" (Daniel, 1990, p. 365).

Research Questions 4 and 5:

4. What is the nature of students' difficulties?
5. Do teachers' speculations about the nature of students' difficulties correspond to actual student difficulties as identified from student performance on the MDT and interviews?

For Research Questions 4 and 5, students' and teachers' written comments from the Research Questionnaires were coded and analyzed, and semi-structured interviews (Clement, 2000; Goldin, 1999) were used with both students and teachers. The choice of the "think-aloud" procedure was justified by: 1) the need for probing questions to be posed at the interviewer's discretion; and 2) the opportunity to gain access to students' and teachers' thought processes as they occur. Open coding techniques and procedures described by Strauss and Corbin (1998) were used to code students' and teachers' answers from Research Questionnaires and Interviews, and develop Preliminary Coding Categories. Of note is the convergent approach of the type of analyses performed. Replicability of observations was desired, and therefore Preliminary Coding Categories were refined and developed into Final Coding Categories (see Appendix B), and raters were trained and retrained until satisfactory inter-rater agreement was achieved (see Chapter 4).

Chapter Summary. In this chapter, the instruments adapted or developed particularly for use in the study were described along with their scoring, and

included: the MDT, the main instrument for measuring student performance on linearity and linear functions, the accompanying Scoring Guide for scoring the MDT, Ranking and Information Questionnaires, and Interviews, together with Preliminary Coding Categories for RQ written comments and Interviews. Characteristics of the sample and the general population were presented, together with the timeline for the design and administration of the instruments, and data analyses. A presentation of the methods of analyses used to answer the research questions concluded the chapter. In the next chapter, Chapter 4, data analyses and results of analyses are presented.

Chapter 4

DATA ANALYSES

4.1 Introduction

The goals of this study were to gain insight into Grades 8, 9, and 10 students' difficulties with concepts of linearity and linear functions and mathematics teachers' understanding of the nature of those difficulties. Participants included 1,561 students (757 Grade 8 students, 416 Grade 9 students, and 388 Grade 10 students), and their 26 current mathematics teachers. Three assessment tools were administered to all participants in this study: Information Questionnaire (IQ), Mini-Diagnostic Test (MDT), and Ranking Questionnaire (RQ). The Information Questionnaire for Students (IQS) requests demographic information, including current mathematics course. The Information Questionnaire for Teachers (IQT) requests demographic information, including current and recently taught mathematics courses, undergraduate college major, and most advanced academic degree, and number of years of teaching. The Mini-Diagnostic Test (MDT) consists of nine problems (seven settings; one problem has three parts) focusing on aspects of linearity and linear functions. The Ranking Questionnaire for Students (RQS) requires the ordering of MDT problems from most to least difficult with a description of the nature of the difficulty for the problem identified as most difficult. The Research Questionnaire for Teachers (RQT) requests the ordering of MDT problems by difficulty from the perspective of their students with descriptions about the nature of the difficulties for the two problems identified as most and next most difficult. After the IQ, MDT, and RQ were

administered, 40 students and 20 teachers were interviewed. Data were collected to answer the following questions:

1. Does student performance on the Mini-Diagnostic Test (MDT) vary by:
    a. Grade level?
    b. Current mathematics course?
    c. Problem?
2. Does student ranking of problems by difficulty, after completing the MDT, correspond to their performance on the MDT?
3. Does teacher ranking of problems by difficulty, after completing the MDT, correspond to student performance on the MDT?
4. What is the nature of students' difficulties?
5. Do teachers' speculations about the nature of students' difficulties correspond to actual student difficulties as identified from student MDT performance and interviews?

In this chapter the data and methods of analyses employed to answer the questions and results of analyses are reported by research question. For Research Questions 1, 4 and 5 inter-rater agreement analyses are addressed first. For Research Question 1, the reliability of the instrument (MDT) is reported as well.

4.2 Research Question 1

Research Question 1: Does student performance on the Mini-Diagnostic Test (MDT) vary by:

a. Grade level?

b. Current mathematics course?

c. Problem?

4.2.1 Inter-Rater Agreement and Instrument Reliability. All MDTs (N=1561) were scored by the researcher using the Scoring Guide (see Tables 1-7 in Chapter 3, and Appendix B). To establish the reliability of the Scoring Guide, 59 MDTs administered to Grade 9 students enrolled in Algebra I or Geometry were scored by a second rater. An inter-rater agreement analysis using the Kappa statistic (Cohen, 1960) was performed to determine consistency among raters for the scoring of the types of student answers to the MDT problems with short or extended answers (problems 1a, 1b, 2, 4, 7a, 7b, and 7c). MDT problems 3, 5, and 6, with multiple-choice answers, were excluded from the inter-rater agreement evaluation.

The following assumptions were made when calculating the inter-rater agreement (Cohen, 1960):

i) The units of analysis representing students' answers on MDT are independent.

ii) The categories represented by student difficulties with MDT problems 1a, 1b, 2, 4, 7a, 7b, 7c are of nominal scale, independent, mutually exclusive and exhaustive.

iii) The raters operate independently, and are equally competent.

The inter-rater reliability of scoring was measured using Cohen's Kappa (*k*) coefficient of agreement representing the proportion of agreement beyond chance agreement:

$$k = \frac{p_0 - p_c}{1 - p_c},$$

where p_0 = the proportion of units in which the raters agreed, and p_c = the proportion of units for which agreement is expected by chance (Cohen, 1960, pp. 19-20).

The inter-rater agreement for the two raters is presented in Table 19.

Table 19

Inter-Rater Agreement on Scoring of Responses

MDT Problem	Observed Kappa(*k*)	95% Confidence Interval	
		Lower Limit	Upper Limit
Problem 1a	.82	.48	1
Problem 1b	.83	.64	1
Problem 4	.86	.66	1
Problems 2, 7a, 7b, 7c	1	1	1

As can be seen in Table 19, the inter-rater agreement for MDT problems 2, 7a, 7b, and 7c is perfect ($k = 1$). This was expected, given the fact that the types of answers for MDT problem 2 are based on the number of rows in the table that are completed correctly, and all "Incorrect answers" are grouped together for MDT problems 7a, 7b, and 7c. For MDT sub-problems 1a, 1b, and MDT problem 4, in which incorrect responses are separated into several types, the inter-rater agreement is high ($k > .82$). The lowest value of inter-rater agreement is for sub-problem 1a, with $k = .82$ ($p < .05$), 95% CI (.48, 1).

For data collected for this study (N=1561), the reliability of the MDT, seen as internal consistency of the test, was measured using Cronbach's Alpha. The coefficient Alpha is a measure of the homogeneity of the test, i.e., if the MDT problems taken together measure the construct referring to Algebra I student knowledge of linearity and linear functions. The value of Alpha was .76, greater than .7, which generally is accepted as cut-off value in the exploratory phase of instrument development (Henson, 2001).

Data were analyzed using descriptive statistics, including cluster analysis and factor analysis. A regression framework was not possible because of the nested structure of the data, i.e., student observations nested within teachers, and thus the violation of the independence assumption necessary for regression. A hierarchical linear model was not applied because of insufficient data at Level 1 (i.e., student demographics).

4.2.2 Student Performance on the MDT by Grade Level. To see how student performance on the MDT varies by grade level, the student MDT total scores were analyzed by computing means and standard deviations by grade level. Table 20 shows the means and standard deviations for the MDT scores by grade level. The MDT has nine problems. Each problem has scores between 0 and 1, thus the MDT total score ranges from 0 through 9. To better illustrate the student MDT score distribution by grade, the corresponding number of students and percentage are included.

Table 20

MDT Performance by Grade Level

Grade	Number of Students (Percentage of Total N, N=1561)	Mean	Standard Deviation
8	757 (48.5%)	4.78	2.27
9	416 (26.6%)	5.20	2.50
10	388 (24.9%)	5.45	2.44
All Students	1561 (100.0%)	5.06	2.39

As can be seen in Table 20, about one-half (48.5%) of the participating students were in Grade 8 and about one-fourth (26.6%, 24.9%) in each of the other grades, 9 and 10 respectively. The means for the MDT score varied little by grade level, ranging from 4.78 for Grade 8 students to 5.45 for Grade 10 students. Overall, the mean for the MDT was 5.06. Regardless of grade level, students

correctly solved five of the nine MDT problems, on average. The standard deviation was about the same for all grades.

To facilitate visualization of the distribution of the MDT scores by grade level, Figures 1a, 1b, and 1c present the data in histograms.

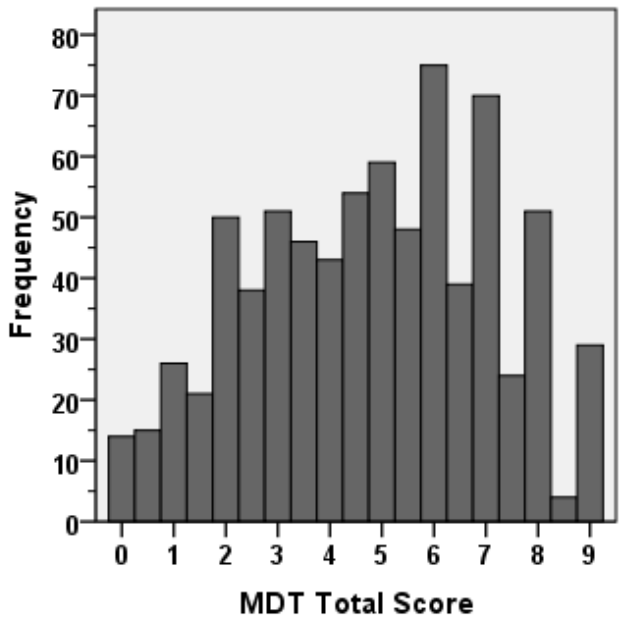


Figure 1a. Histogram of MDT Scores Distribution for Grade 8 Students

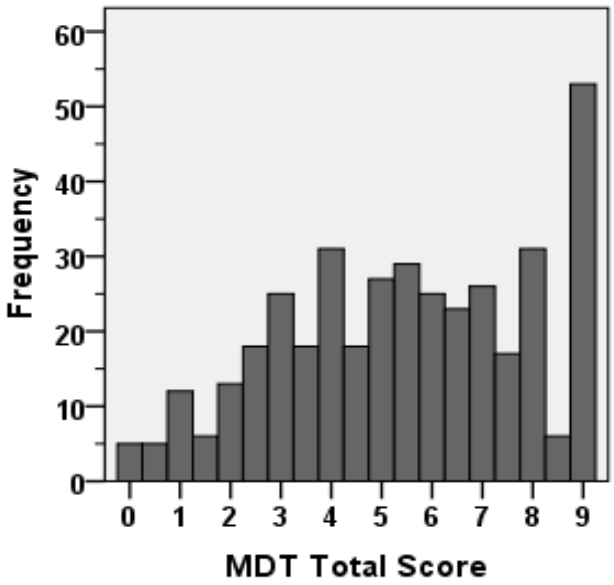


Figure 1b. Histogram of MDT Scores Distribution for Grade 9 Students

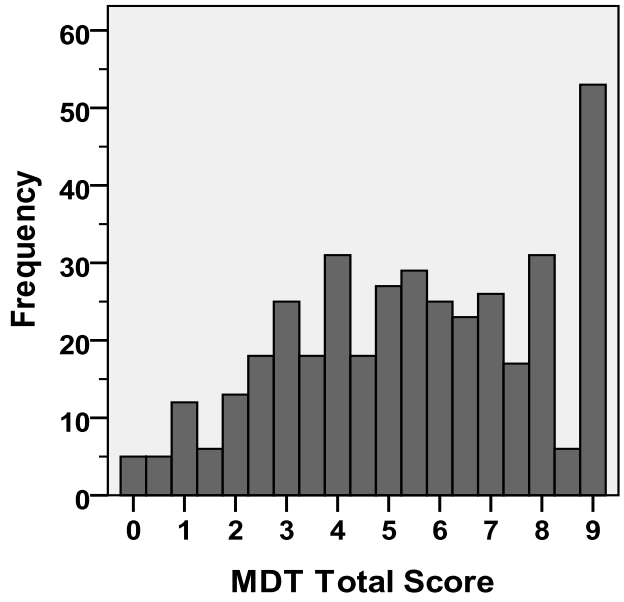


Figure 1c. Histogram of MDT Scores Distribution for Grade 10 Students

As can be seen in the histograms, MDT scores ranged from 0 through 9 with a mean of about 5 for all grades. For Grade 8 (Figure 1a), the mode was 6. For Grade 9 (Figure 1b) and Grade 10 (Figure 1c), the mode was 9, the maximum score possible. It is easily noticeable in Figures 1a, 1b, and 1c that the distributions are asymmetric due to the greater number of higher scores. More than 50 students in Grades 9 and 10 had MDT scores of 9.

4.2.3 Student Performance on the MDT by Current Mathematics Course. Table 21 presents the means and standard deviations of MDT scores by current mathematics course (Pre-Algebra through Algebra II), accompanied by the number of students and corresponding percentage of participating students enrolled in those courses.

Table 21

MDT Student Performance by Current Mathematics Course

Current Mathematics Course	Number of Students (Percentage of Total N, N=1561)	Mean	Standard Deviation
Pre-Algebra	457 (29.3%)	3.75	1.98
Algebra I	575 (36.8%)	5.13	2.17
Geometry	452 (29%)	5.74	2.38
Algebra II	77 (4.9%)	8.36	0.97
All Students	1561 (100.0%)	5.06	2.39

As can be seen in Table 21, there were about the same number of students enrolled in Pre-Algebra (N=457) as in Geometry (N=452). Algebra I had the

highest enrollment (N=575), and Algebra II, an advanced course usually taken by students in Grade 11, had the lowest enrollment (N=77). Pre-Algebra students scored lowest on the MDT, correctly solving, on average, about four problems. Mean scores for Algebra I and Geometry students were about the same; on average, those students correctly solved about five problems. The standard deviation for Pre-Algebra, Algebra I, and Geometry was about the same. Two thirds of Algebra II students correctly solved all nine problems. The standard deviation of the MDT scores for Algebra II students was less than 1, the least for all groups of students.

Figures 2a, 2b, 2c, and 2d present histograms of the MDT score distributions for students enrolled in Pre-Algebra, Algebra I, Geometry, and Algebra II, respectively.

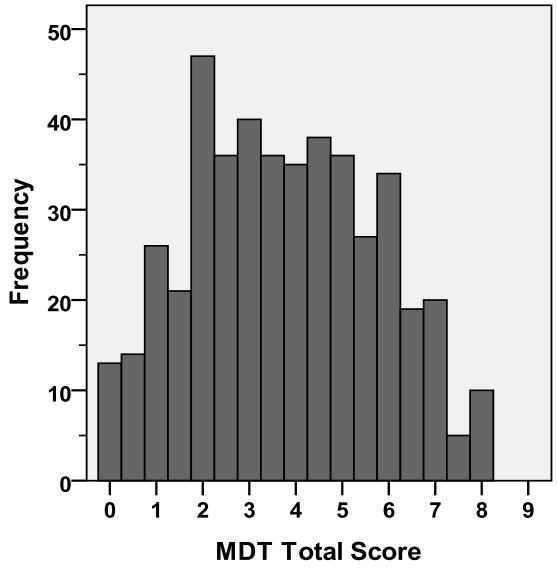


Figure 2a. Histogram of MDT Scores Distribution for Pre-Algebra Students

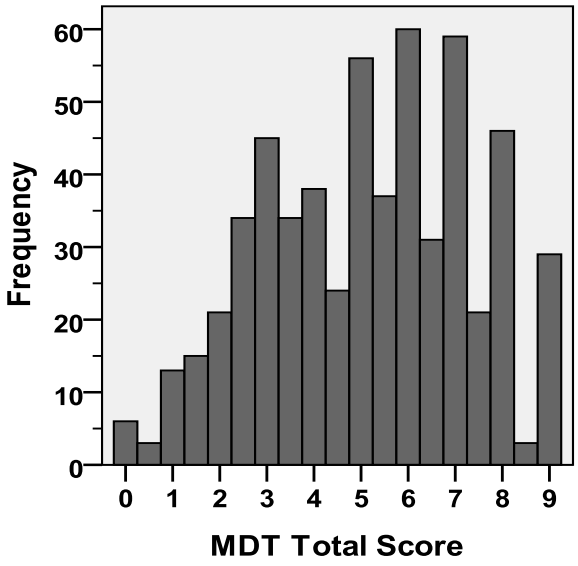


Figure 2b. Histogram of MDT Scores Distribution for Algebra I Students

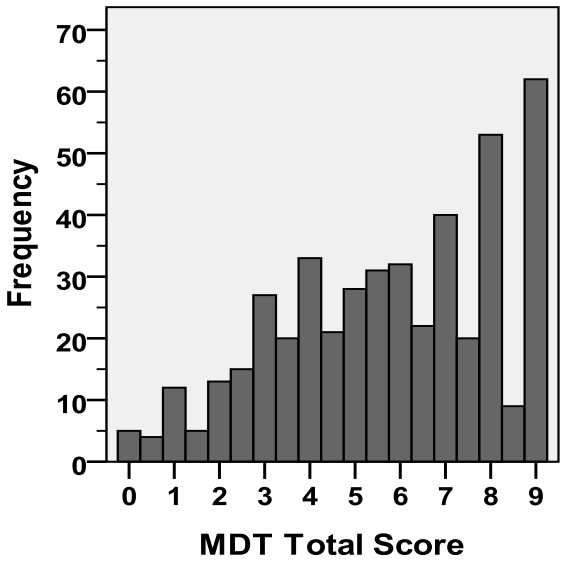


Figure 2c. Histogram of MDT Scores Distribution for Geometry Students

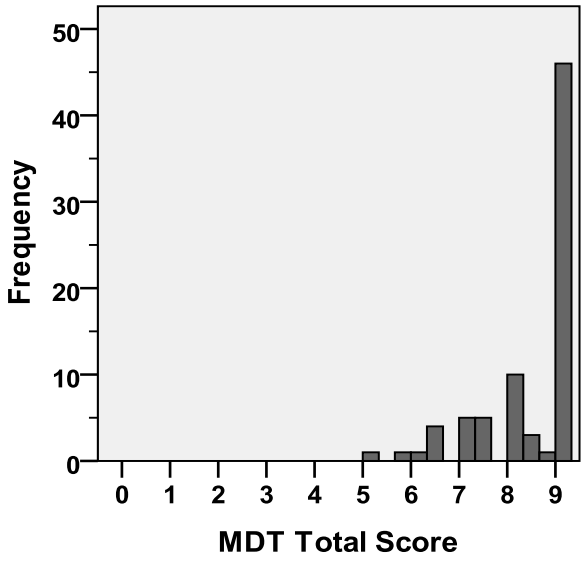


Figure 2d. Histogram of MDT Scores Distribution for Algebra II Students

Figures 2a, 2b, 2c, and 2d show that the MDT score distributions are symmetric for Pre-Algebra students, and become increasingly skewed for students enrolled in higher level courses, Algebra I, Geometry, and Algebra II. Higher MDT scores were associated with higher-level mathematics courses.

4.2.3.1 Algebra I Students' Performance on the MDT by Grade. Algebra I students' performance on the MDT was central to this study, since Algebra I is the course in which students are expected to master aspects of linearity and linear functions (Arizona Department of Education, 2008; CCSSO & NGA, 2010). To see how Algebra I students' performance on the MDT varies by grade, Table 22 presents the means and standard deviations for MDT scores by grade level for students enrolled in Algebra I. The second column of the table shows the number and corresponding percentage of students in each grade who were enrolled in Algebra I.

Table 22

Algebra I Students' Performance on the MDT by Grade

Grade	Number of Students (Percentage of Total N, N= 575)	Mean	Standard Deviation
8	286 (49.7%)	6.33	1.72
9	281 (48.9%)	3.97	1.89
10	8 (1.4%)	2.95	1.66
All Algebra I Students	575 (100%)	5.13	2.17

As can be seen in Table 22, about half of the Algebra I students who completed the MDT were in Grade 8 (49.7%) and half were in Grade 9 (48.9%). Means on the MDT decreased as grade level increased. On average, Grade 8 students correctly solved about six MDT problems; Grade 9 students correctly solved, on average, about four MDT problems; and Grade 10 students correctly solved, on average, three MDT problems. On average, Algebra I students (all grades) correctly solved about five MDT problems. Standard deviations were about the same for each grade level.

4.2.4 MDT Student Scores by Performance Group. Data collected led to consideration of three possible natural clusters for grouping students' MDT scores: by grade level, current mathematics course, and performance group. There were two performance groups, Group 0 and Group 1, described in detail in Chapter 3. Group 0 consisted of students enrolled in mathematics courses at or below their grade level. Group1 consisted of students enrolled in more advanced mathematics courses, that is, courses above their grade level. A cluster analysis revealed that MDT scores were better grouped (more cohesion, less separation) by student performance group than by grade or current mathematics course. The average silhouette criterion (Rousseeuw, 1987) was used for determining the best cluster. The largest overall average silhouette indicates the best clustering (Rousseeuw, 1987).

Figures 3a, 3b, and 3c illustrate a comparison of the quality of clusters, using the average silhouette as a criterion for determining the best cluster from three

possible natural clusters, by grade, current mathematics course, and performance group, respectively.



Figure 3a. Silhouette Measures of Cohesion and Separation – Clustering by Grade



Figure 3b. Silhouette Measures of Cohesion and Separation – Clustering by Current Mathematics Course

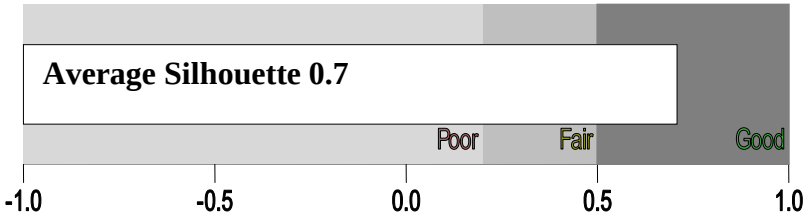


Figure 3c. Silhouette Measures of Cohesion and Separation – Clustering by Performance Group

As can be seen in Figures 3a, 3b, and 3c, the average silhouette for the three clusters by grade, current mathematics course, and performance group are .6, .5, and .7, respectively. The best cluster is the last one, by performance group, with a good average silhouette of .7.

Table 23 presents the means and standard deviations of the MDT scores by performance level. The first column of the table shows the number and corresponding percentage of students for each performance level.

Table 23

MDT Scores by Performance Level

Performance Level	Number of Students (Percentage of Total N, N=1561)	Mean	Standard Deviation
Group 0 (Grade Level or Below)	1021 (65.4%)	4.06	2.03
Group 1 (Above Grade Level)	540 (34.6%)	6.95	1.83
All Students	1561 (100.0%)	5.06	2.39

As Table 23 shows, 65.4% of all students were from Group 0. On average, Group 0 students correctly solved about four MDT problems with a standard deviation of about 2. On average, Group 1 students, representing 34.6% of the total number of students, correctly solved about seven MDT problems with a standard deviation of about 2.

Figures 4a and 4b show histograms of the MDT score distribution by performance level for students in Group 0 and Group 1, respectively.

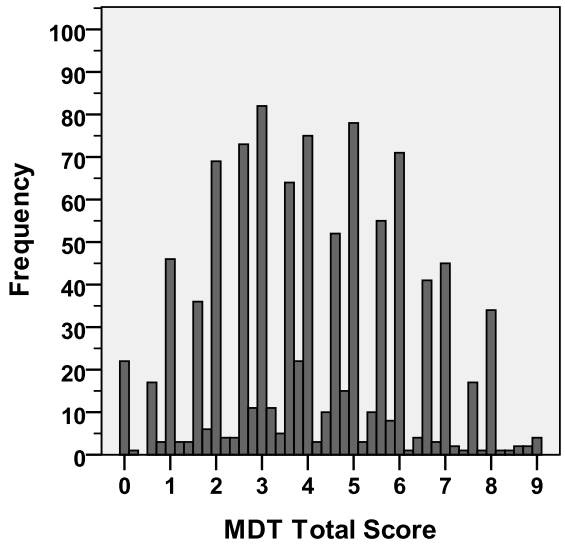


Figure 4a. Histogram of MDT Score Distribution for Group 0 Students

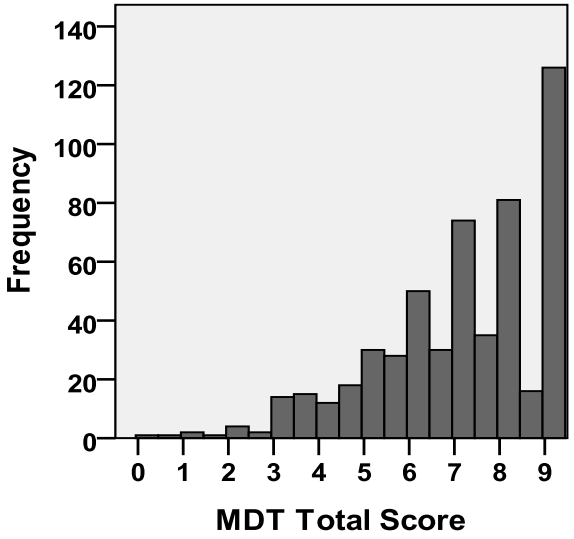


Figure 4b. Histogram of MDT Score Distribution for Group 1 Students

Figures 4a and 4b show that the MDT score distribution for Group 0 is fairly symmetric, while the one for Group 1 is heavily skewed due to the presence of many high MDT scores.

4.2.5 Student Performance on the MDT by Problem. To gain insight into the variation of student performance on the MDT by problem, the nine MDT problem scores were used to perform an Exploratory Factor Analysis (EFA), R-type, using the Principal Component Analysis method (Hair et al., 1998).

4.2.5.1 Exploratory Factor Analysis. The EFA was performed to:

i) Reduce the original set of variables (problem scores) to a new set with a fewer number of variables (factors) that could explain the maximum of the variance of the original set; and

ii) Detect the structure of relationships among the variables representing the student scores on the MDT problems.

After the data were reduced, the new reduced data were used in all subsequent analyses performed to answer Research Questions 4 and 5.

Data Types and the Choice of the Statistical Software. The variables representing the scores for the MDT problems can be considered ordinal under the assumption that a higher score for a problem corresponds to a higher level of student performance. All variables representing the scores for the MDT problems can be considered dichotomous or polytomous. The variables representing the

scores for MDT problems 3, 4, 5, 6, 7a, and 7b have two values (0 or 1). The variables representing the scores for MDT problems 1 and 7c have three values (0, .5, or 1). The variable representing the score for problem 2 has six values (0, .2, .4, .6, .8, or 1). There is growing consensus that the association between dichotomous and polytomous variables is better expressed by tetrachoric or polychoric correlations (Holgado–Tello, Chacón–Moscoso, Barbero–García, & Vila–Abad, 2008). "The polychoric correlation estimates the linear relationship between two unobserved continuous variables given only observed ordinal data" (Flora & Curran, 2004, p. 467). The tetrachoric correlation is a particular case of the polychoric correlation between two dichotomous variables. "[It] estimates the Pearson correlation one would obtain if the two constructs were measured continuously" (Uebersax, 2000). The Exploratory Factor Analyses (EFA) were performed using MicroFACT2.0, appropriate for tetrachoric and polychoric correlations (Uebersax, 2006; Yaffee, 2003).

About the Sample. The sample size is large (N=1561), suitable for EFA (Costello & Osborne, 2005). It exceeds the criterion of a minimum of 20 observations per variable (Hair et al., 1988).

Table 24 shows the means and standard deviations for the scores for MDT problems 1, 2, 3, 4, 5, 6, 7a, 7b, and 7c.

Table 24

Means and Standard Deviations for Scores on MDT Problems

Problem	Mean	Standard Deviation
1	.51	.46
2	.63	.45
3	.78	.41
4	.35	.48
5	.81	.39
6	.51	.50
7a	.59	.49
7b	.29	.45
7c	.57	.41

As can be seen in Table 24, there is no mean value indicative of outliers (greater than .9 or less than .1) which might create floor or ceiling effects (McBee, 2010) and require the elimination of the respective variable from analysis. Floor or ceiling effects occur when problems are too difficult, or too easy, thus making it more difficult to detect true differences in student performance. The most difficult MDT problems, with respect to student performance, were 7b and 4, having the lowest means of .29 and .35, respectively. Both problems 7b and 4 have short answer formats and require calculation of the slope of a line given the graph of the line. The least difficult MDT problems, with respect to student performance, were problems 5 and 3, with the highest means of .81 and .78, respectively. MDT problems 5 and 3 have multiple-choice formats, and require making connections among representations. The standard deviations

for all nine scores of the MDT problems ranged from .39 to .5, with little variation.

Assumptions and the Correlation Matrices. The basic assumptions, i.e., normality, homoscedasticity, and linearity are "more conceptual than statistical" (Hair et al., 1998, p. 99) and necessary as long as their violation might lower the observed correlations. "Only normality is necessary if a statistical test is applied to the significance of the factors" (Hair et al., p. 99), but normality is improbable in education and psychology, especially for the type of large sample (N=1561) in the current study (Micceri, 1989). As such, the appropriateness of the EFA was assessed by examining the matrices of correlation among variables.

Table 25 shows the teatrachoric and polychoric correlations among the scores on the nine MDT problems 1, 2, 3, 4, 5, 6, 7a, 7b, and 7c.

Table 25

Correlations among Scores on the MDT Problems (MicroFACT)

Problem	1	2	3	4	5	6	7a	7b	7c
1	1.00	.56	.46	.46	.31	.26	.37	.45	.41
2		1.00	.57	.49	.44	.30	.48	.45	.51
3			1.00	.37	.42	.13	.41	.36	.45
4				1.00	.32	.43	.46	.55	.44
5					1.00	.09	.35	.32	.40
6						1.00	.37	.40	.29
7a							1.00	.50	.59
7b								1.00	.65
7c									1.00

Table 25 shows that most of the correlations are greater than .3 (rounded), therefore the appropriateness of EFA is justified. The minimum correlation (.09) is between MDT multiple-choice problem 5 (finding the linear function given a table of values) and problem 6 (finding the *y*-intercept given the equation of a line in standard form). The maximum correlation (.65) is between MDT problem 7b (finding the slope given the graph of a line) and problem 7c (finding the speed given the graph representing the change in distance as a function of change in time). There is also a high correlation (.59) between MDT problem 7a (comparing speeds at three intervals of time, given the graph representing the change in distance as a function of change in time) and MDT problem 7c. Given that MDT problems 7a, 7b, and 7c refer to the same situation, higher correlations might have been theoretically justified. However, those correlations express the empirical

relationships in the data, how the students thought about those problems, and not theoretical relationships. Also high are the correlations .56 and .57 for MDT problems 1 and 2, and problems 2 and 3, respectively. MDT problems 1, 2, and 3 illustrate the Cartesian connection.

To assist in determining if any of the problems might have been dropped from analysis, correlations of the MDT problems with the MDT total score were calculated. Table 26 shows those correlations.

Table 26

Correlations of Problems with MDT Total Score

Problem	1	2	3	4	5	6	7a	7b	7c
Correlation with MDT Total Score	.62	.78	.78	.66	.73	.59	.79	.65	.70

In Table 26, all correlations are high, ranging from .6 to .8 (rounded). Therefore no variable was dropped from analysis.

As mentioned before, the internal consistency of the MDT was determined using Cronbach's Alpha as a measure of the correlation between observed and true scores. The test was found reliable, with a Cronbach's Alpha value of .76, which is greater than .7, the accepted cut-off value (Henson, 2001).

Factors and their Interpretation. After the extraction of two initial factors, trials with three factors were performed as well. The goal was to extract an optimum number of factors that would lead to the simplest possible structure

while accounting for the largest percentage of the explained variance. More factors extracted account for higher percentages of explained variance but lead to more complicated structures, while fewer factors extracted lead to simpler structures but account for smaller percentages of the explained variance. The criteria applied when determining the number of factors to extract (Hair et al., 1998) were the following:

i. *Kaiser (Latent Root, Eigenvalue) Criterion* (Hair et al., 1998, pp. 103-104) is the criterion used most often. Since each variable accounts for a value of 1 from the total eigenvalue, only the factors with eigenvalues greater than 1 that can account for at least one variable are considered significant. This criterion was applied for determining the initial number of factors extracted. Table 27 shows the initial eigenvalues for each of the nine components.

Table 27

Initial Eigenvalues

Component	1	2	3	4	5	6	7	8	9
Initial Eigenvalues	4.35	1.09	0.76	0.63	0.57	0.48	0.45	0.38	0.29

As can be seen in Table 27, there are two eigenvalues greater than 1, (4.35 and 1.09). Therefore, according to *Kaiser Criterion*, two factors for the 9-variable set should be extracted.

ii. *Scree Test Criterion* (Hair et al., 1998, pp. 104-105) uses the plot (line graph) of the eigenvalue for each factor to determine the cut-off point where the segment lines are almost horizontal (the slopes of the segment lines approach 0). The factors that account for most of the variance are those selected before the cut-off point. Figure 5 represents the scree plot of the initial eigenvalues from Table 27.

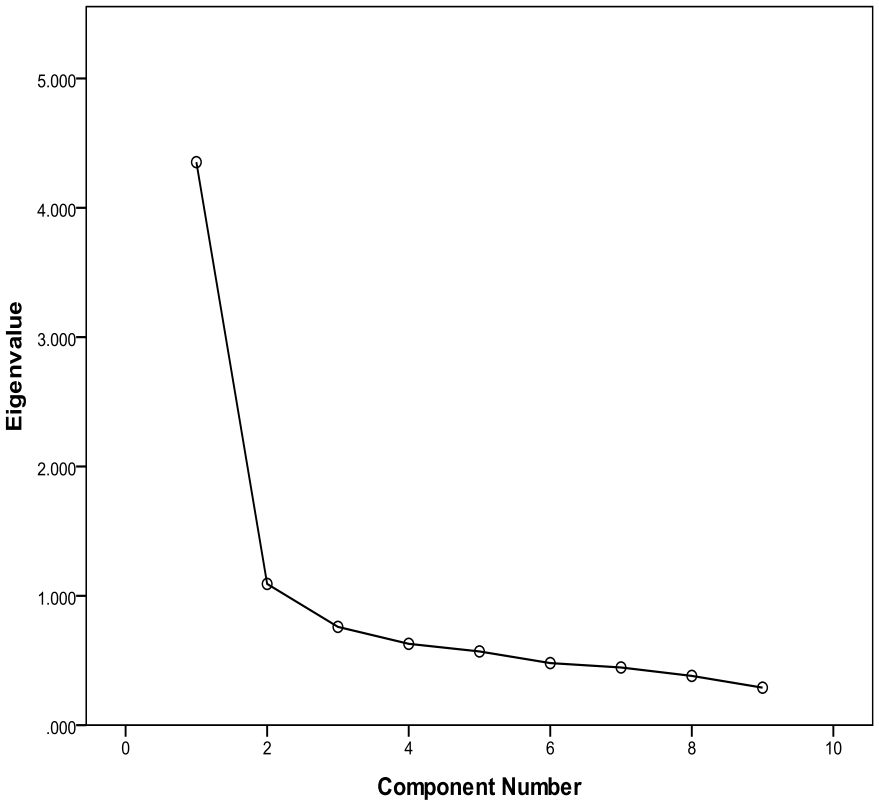


Figure 5. Scree Plot of the Initial Eigenvalues

Figure 5 shows that only the first line segment has a slope with absolute value greater than 1, while all other line segments are less and less steep, with slopes with absolute values between 0 and 1.

According to the *Scree Test Criterion*, two factors account for most of the variance, therefore two factors should be extracted.

iii. *Percentage of Variance Criterion* (Hair et al., 1998, p. 104) takes into consideration the percentage of total explained variance acceptable for research (above 95% in natural sciences, 60% and below in social sciences), and extracts the number of factors accounting for this a priori established amount of cumulative variance.

According to this criterion, a two-factor solution accounts for 49% of the explained cumulative variance, a moderate, acceptable value in educational studies (Hair et al., 1998).

iv. *Comprehensibility Criterion* aims to ensure the parsimony of the model, and therefore restricts the number of factors extracted, usually to two or three. At the same time, a factor that accounts for only one or two variables is weak. Therefore it is desirable to have factors that account for at least three variables. Since there are nine variables, the number of factors is restricted to a maximum of three. Trials with three factors revealed that the third factor was a weak one, accounting for at most two variables. Table 28 shows the rotated factor loadings for three factors. The VARIMAX rotation method is used for easier interpretation of the

factors. With the VARIMAX method, high loadings are expected in each column, close to 1 or -1, indicating associations between variable and factor, and small loadings, close to 0, indicating lack of association between variable and factor (Hair et al., 1998, p. 110).

Table 28

VARIMAX Rotated Factor Loadings (Three Factors)

Problem	Factor 1	Factor 2	Factor 3
1	.56	.16	.33
2	.73	.21	.30
3	.69	.24	.08
4	.39	.25	.56
5	.49	.28	.05
6	.06	.14	.68
7a	.37	.46	.37
7b	.29	.54	.49
7c	.35	.80	.23

Table 28 indicates that MDT problems 1, 2, 3, and 5 loaded significantly on the first factor. MDT problems 7a, 7b, and 7c loaded significantly on two or three factors, making the interpretation difficult. MDT problems 4 and 6 loaded on the third factor.

According to *Comprehensibility Criterion*, and given the small number of variables, a solution with two factors seems preferable (Hair et al., 1998, pp.106-113).

v. *Heterogeneity of the Respondents Criterion* (Hair et al., 1998, p. 105) is useful in situations when the researcher suspects that the sample is not homogenous, and more factors may be purposefully extracted. It is expected that factors reflecting the variables that are more homogeneous across the sample will be extracted first, while later factors will be loaded with variables discriminating across the sample.

Table 29 shows the means of scores of the nine MDT problems for all students (Overall) and for students from Group 0 and Group 1.

Table 29

Means of Scores of MDT Problems by Group of Students

Problem	1	2	3	4	5	6	7a	7b	7c
Overall	.51	.64	.78	.35	.81	.51	.59	.29	.57
Group 0	.39	.51	.69	.24	.76	.41	.47	.16	.44
Group 1	.76	.90	.95	.58	.91	.71	.81	.53	.81

As can be seen in Table 29, Group 1 outperformed Group 0. Indeed, all Group 1 means are greater than Group 0 means. For example, for MDT problems 7b and 4 with the lowest overall means, .29 and .35, respectively, there were more than three times (.53/.16) as many students in Group 1 than in Group 0 who correctly solved MDT problem 7b, and more than twice (.58/.24) as many students in Group 1 than in Group 0 who correctly solved MDT problem 4. Since more than 90% of Group 1 students correctly solved MDT problems 2, 3, and 5, these variables would produce outliers and should be dropped from analysis.

According to *Heterogeneity of the Respondents Criterion*, more factors should be extracted. In this study the desirable number of factors is three, but the number of factors that could be extracted is restricted to two, given the small number of variables.

vi. *Summary of Factor Selection Criteria*. The prevailing criterion is to obtain a simple structure where each variable loads highly on only one factor. Trials with two factors and then three factors were performed. Weak factors, with high loadings for only one or two variables were excluded, which limited the analysis to two-factor analysis. Table 30 shows the factor loading and final communalities estimates when two factors are extracted.

Table 30

Unrotated Factor Loadings

Problem	Factor 1	Factor 2	Final Communality Estimates
1	.63	.10	.41
2	.75	.23	.62
3	.65	.39	.57
4	.68	-.18	.50
5	.51	.25	.32
6	.45	-.38	.34
7a	.69	-.10	.48
7b	.74	-.28	.62
7c	.74	-.06	.55

The columns labeled Factor 1 and Factor 2 in Table 30 show the factor loadings (correlations between variables and the two extracted factors) for each of the nine MDT problems. All variables, except MDT problem 6, loaded highly (> .5) on the first factor and lowly on the second factor (< .4), which indicates homogeneity of the sample with respect to those variables. Of note, regarding MDT problem 6, interviews with students and their teachers from one of the middle schools revealed that Pre-Algebra students did not learn about finding the *y*-intercept of a line when given the linear equation $ax + by = c$. The final communality estimates (last column) indicates how much variance in a variable is accounted for by the two-factor solution. Note that a high communality of a variable indicates the reliability of the factor analysis, i.e., the variable is well represented by the extracted factors. Low communalities are acceptable if the sample size is large, and in this case the sample size is large (N=1561), allowing for lower values (< .4) of communalities to be acceptable. The total percentage of trace (obtained by adding the final communality estimates and dividing the sum by the number of variables) is 49%, a moderate value, which shows how different the variables are from one another. Small communalities for MDT problems 1, 5, 6, and 7a indicate that more than half of the variance in those variables remains unaccounted for by the two- factor solution.

The total variance explained by these two factors is split unevenly between factors: Factor 1 accounts for 43%, and Factor 2 accounts for only 6% of the total variance. If the factors are rotated, the same amount of variance is explained, but is distributed more evenly between factors, 25.6% and 23.4%, respectively. The

aim is to obtain a matrix with either high loadings (close to 1 or -1), or low values (close to 0) in each column of the factor matrix. Therefore a rotation was performed using the VARIMAX method.

Table 31 shows the VARIMAX rotated factor matrix.

Table 31

VARIMAX Rotated Factor Loadings (Two Factors)

Problem	Factor 1	Factor 2
1	.53	.36
2	.71	.35
3	.74	.16
4	.37	.60
5	.54	.18
6	.06	.58
7a	.43	.55
7b	.35	.71
7c	.49	.55

As can be seen in Table 31, the rotated factor matrix offers a simple structure that can be easily interpreted. MDT problems 1, 2, 3, and 5 loaded significantly on Factor 1 (factor loadings > .5). Conceptually, all of those problems are situated across the dimension of connections among representations. Indeed, MDT problem 1 requires making the Cartesian connection from a point to the equation of the line. MDT problem 2 requires making the Cartesian connection from the equation of a line to points on the line. MDT problem 3 requires making the

connection between a table of values and a graphic representation of a linear function. MDT problem 5 requires connecting a description of a real-life situation to its symbolic representation, a linear function. Given these theoretical considerations, Factor 1 was labeled *Connections among Representations*.

MDT problems 4, 6, 7a, 7b, and 7c loaded significantly on Factor 2 (factor loadings > .5). Conceptually, the corresponding MDT problems require determining the slope of a line from the graph of the line (MDT problems 4, and 7b), finding the *y*-intercept of a line given the equation of the line in standard form (MDT problem 6), finding the rate of change/speed of a car given the linear graph of the distance-time function (MDT problem 7c), and comparing three speeds of a car given the three-segment linear graph of the distance-time function (MDT problem 7a). Conceptually, MDT problems 4, 6, 7a, 7b, and 7c require the identification and interpretation of the rate of change/slope and the *y*-intercept of a linear function in various contexts. Therefore Factor 2 was labeled *Linear Functions.*

Table 32 illustrates the structure obtained using the two-factor analysis.

Table 32

Labeled Factor Matrix

Problem	Factor 1	Factor 2
	Connections among Representations	*Linear Functions*
1	.53	
2	.71	
3	.74	
4		.60
5	.54	
6		.58
7a		.55
7b		.71
7c		.55

As can be seen in Table 32, MDT problems 1, 2, 3, and 5 were grouped along the *Connections among Representations* dimension, and MDT problems 4, 6, 7a, 7b, and 7c were grouped along the *Linear Functions* dimension. To better reveal the structure of relationships between variables for each of the factors, all low loadings (< .5) were suppressed.

Validation of Factor Analysis. Exploratory factor analyses were performed separately for Group 0 and Group 1 students.

Table 33 and Table 34 present the rotated factor loadings, using the VARIMAX method, for Group 0 and Group 1, respectively.

Table 33

Rotated Factor Loadings for Group 0

Problem	Factor 1	Factor 2
	Connections among Representations	*Linear Functions*
1	.48	
2	.72	
3	.64	
4		.48
5	.50	
6		.33
7a		.53
7b		.72
7c		.51

Table 34

Rotated Factor Loadings for Group 1

Problem	Factor 2	Factor 1
	Linear Functions	*Connections among Representations*
1		.81
4	.71	
6	.60	
7a	.51	
7b	.56	

The structure of relationships for Group 0 presented in Table 33 is identical with the one representing the structure of relationships for the entire sample presented in Table 32. Indeed, MDT problems 1, 2, 3, and 5 were grouped along the *Connections among Representations* factor, and MDT problems 4, 6, 7a, 7b, and 7c were grouped along the *Linear Functions* factor.

As can be seen in Table 34, MDT problems 2, 3, 5, and 7c were deleted from analysis because they had mean scores above .9 and produced outliers. Notwithstanding the reduction in the number of variables, the relationships between variables remained the same for both groups of students, i.e., the variables common to both groups were associated in the same way.

Surrogate Variables for Data Reduction. The method used for data reduction was the selection of surrogate variables to represent the entire set of variables. A surrogate variable is a variable that represents the entire factor in further analyses. This method of reduction was preferred being easier to interpret, though it left out the other variables. The rotation used for obtaining a simple factor structure was orthogonal. Therefore the factors were independent, allowing for the selection of surrogate variables, uncorrelated if possible (Hair et al., 1998, p. 129).

The magnitude of loading is decisive in selecting the surrogate variable. As such, the following surrogate variables were selected:

a) For Group 0, MDT problem 2, with loading .72 (Table 33), was selected as surrogate variable for the *Connections among*

Representations factor. MDT problem 7b, with loading .72 (Table 33), was selected as surrogate variable for the *Linear Functions* factor.

b) For Group 1, the only choice for the surrogate variable representing the *Connections among Representations* factor was MDT problem 1 with loading .81 (Table 34). MDT problem 4 with loading .71 (Table 34) was selected as surrogate variable for the *Linear Functions* factor.

From a theoretical point of view, the choice of surrogate variables is easy to defend. Both MDT problems 1 and 2 illustrate the Cartesian connection, which is representative of *Connections among Representations*. Both MDT problems 7b and 4 focus on slope, which is representative of *Linear Functions*.

4.2.5.2 Variation of Student Performance on the MDT Representative Problems by Grade and Current Mathematics Course.

After data reduction, the analyses about the nature of student difficulties (Research Questions 4 and 5) considered only student performance on MDT problems 2 and 7b for Group 0 students and MDT problems 1 and 4 for Group 1 of students.

Table 35 presents the means of scores for MDT problems 2 and 7b by grade for Group 0 students.

Table 35

Mean Scores for MDT Problems by Grade for Group 0

	Grade 8 (N=457)	Grade 9 (N=263)	Grade 10 (N=301)
Problem 2	.47	.49	.57
Problem 7b	.11	.14	.24

As can be seen in Table 35, the mean scores for MDT problems 2 and 7b for Group 0 students were about the same, with a very small increase ($\leq .03$) from Grade 8 to Grade 9 , and a small increase ($\leq .1$) from Grade 9 to Grade 10. Problem 7b was solved correctly by only 11% of Grade 8 students, 14% of Grade 9 students, and 24% of Grade 10 students.

Table 36 shows the means of scores for MDT problems 2 and 7b by current mathematics course for Group 0 students.

Table 36

Mean Scores for MDT Problems by Current Mathematics Course for Group 0

	Pre-Algebra (N=457)	Algebra I (N=271)	Geometry (N=293)
Problem 2	.47	.49	.58
Problem 7b	.11	.14	.25

It is easily noticeable that the means of scores in Table 36 are almost identical to those in Table 35. These results are expected since 100% of Grade 8

students were enrolled in Pre-Algebra, 100% of Grade 9 students were enrolled in Algebra I, and 97% of Grade 10 students were enrolled in Geometry. A comparison of student performance on the MDT by current mathematics course for Group 0 students shows little difference between Pre-Algebra and Algebra I, as the mean scores for MDT problems 2 and 7b increased by at most, . 03. There was a small difference between Group 0 students enrolled in Algebra I, and those enrolled in Geometry, but the difference between the mean scores for MDT problems 2 and 7b remained small, at most .11.

Table 37 presents mean scores for MDT problems 1 and 4 by grade for students in Group 1.

Table 37

Mean Scores for MDT Problems by Grade for Group 1

	Grade 8 (N=300)	Grade 9 (N=153)	Grade 10 (N=87)
Problem 1	.68	.80	.94
Problem 4	.46	.70	.77

As can be seen in Table 37, the mean scores for MDT problem 1 increased by grade, from .68 in Grade 8 to .94 in Grade 10. MDT problem 1 was successfully mastered by Grade 10 students with a mean score of almost 1 (.94). Although lower than the mean scores for MDT problem 1, the mean scores for MDT problem 4 also increased by grade, from .46 in Grade 8 to .77 in Grade 10. More

than half (54%) of Grade 8 students and about one-fourth (23%) of Grade 10 students did not solve MDT problem 4.

Table 38 shows mean scores for MDT problems 1 and 4 for students in Group1 by current mathematics course.

Table 38

Mean Scores for MDT Problems by Current Mathematics Course for Group 1

	Algebra I (N=304)	Geometry (N=159)	Algebra II (N=77)
Problem 1	.67	.83	.94
Problem 4	.45	.72	.79

Table 38 indicates that Group 1 mean scores for MDT problems 1 and 4 increased by current mathematics course, with values very close to those values shown in Table 37. This result is not surprising since 95% of Grade 8 students were enrolled in Algebra I, 88% of Grade 9 students were enrolled in Geometry, and 89% of Grade 10 students were enrolled in Algebra II. MDT problem 4 was not solved by 28% of Geometry students, and 21% of Algebra II students from Group 1.

Following is a summary of the results for Research Question 1:

1. There was little variation in student performance on the MDT from grade 8 through Grade 10. Mean scores increased from 4.78 to 5.45, representing, on average, less than one more MDT problem correctly solved by Grade 10 students as compared with Grade 8 students.

2. Student performance on the MDT increased with more advanced courses, from Pre-Algebra to Algebra II. Mean scores more than doubled from 3.75 (Pre-Algebra students) to 8.36 (Algebra II students). Algebra II students correctly solved, on average, more than eight of the nine MDT problems.
3. Group 1 students enrolled in mathematics courses above their grade level outperformed Group 0 students enrolled in courses at or below their grade level. The difference on mean scores was 2.88, representing, on average, about three more MDT problems correctly solved.
4. The most difficult problem was identifying the slope of a line from a graph (MDT problem 4). Student performance on MDT problems 7b and 4 representative of the *Linear Functions* factor, remained very low across grades and current mathematics courses for Group 0 students (mean scores between .11 and .25) and low to moderately high across grades and current mathematics courses for Group 1 students (mean scores between .45 and .79). Student performance on MDT problems 2 and 1, representative of the *Connections among Representations* factor, remained low to moderate across grades and current mathematics courses only for Group 0 students (mean scores between .47 and .58), and increased from moderate to very high values for Group 1 students across grades and current mathematics courses (mean scores between .67 and .95).

4.3 Research Question 2

Research Question 2: Does student ranking of problems by difficulty, after completing the MDT, correspond to their performance on the MDT?

All participating students (N=1561) completed the MDT and Ranking Questionnaire for Students (RQS) described in Chapter 3. The MDTs were scored using the Scoring Guide (SG) described also in Chapter 3, and mean scores by MDT problem were calculated for Group 0 and Group 1 students. Mean scores on MDT problems were arranged from least to greatest; the most difficult MDT problem that was ranked highest had the lowest mean. The Ranking Questionnaires for Students were used to calculate frequencies of responses by MDT problem, and obtain the ranking of MDT problems by difficulty, as perceived by students. The most difficult MDT problem, as perceived by students, was the problem with the highest frequency, i.e., identified by the greatest number of students as being the most difficult. To assess the association between the variables representing the ranking of MDT problems by their mean scores and the ranking of MDT problems by difficulty as perceived by students, the coefficient of correlation Kendall-tau was used (Daniel, 1990).

Tables 39 and 40 present the comparison between student performance on the MDT and student ranking of MDT problems by perceived difficulty, for students in Group 0 and Group 1, respectively. Roman numerals are used for ranking, with "I" representing the most difficult problem. The problem number is followed by, in parentheses, mean scores for the MDT problems in the "Problem (Mean)"

column, and frequencies of RQS responses in the "Problem (Number of Students)" column.

Table 39

Group 0 Performance on the MDT vs. Ranking of Problem Difficulty by Students

Ranking by MDT performance	Problem (Mean)	Ranking by RQS	Problem (Number of Students)
I	4 (.24)	I	1 (323)
II	7 (.38)	II	2 (158)
III	1 (.39)	III	7 (136)
IV	6 (.41)	IV	4 (105)
V	2 (.50)	V	6 (112)
VI	3 (.69)	VI	3 (38)
VII	5 (.76)	VII	5 (16)

As can be seen in Table 39, the most difficult MDT problem for Group 0 students was problem 4 (mean score .24), followed by problem 7 (mean score .38). However, students ranked MDT problem 1 (N=323) as most difficult, followed by MDT problem 2 (N=158). MDT problems 3 and 5 were least difficult, and were identified by Group 0 students as being least difficult for them. Of note, when ranking MDT problems by difficulty, students considered MDT problem 7 as being one problem, therefore for ranking analyses, problem 7 was considered as one problem.

Table 40

Group 1 Performance on the MDT vs. Ranking of Problem Difficulty by Students

Ranking by MDT performance	Problem (Mean)	Ranking by RQS	Problem (Number of Students)
I	4 (.58)	I	7 (187)
II	6 (.71)	II	2 (96)
III	7 (.74)	III	4 (82)
IV	1 (.76)	IV	6 (81)
V	2 (.89)	V	1 (33)
VI	5 (.91)	VI	3 (21)
VII	3 (.95)	VII	5 (18)

As shown in Table 40, the most difficult MDT problem for Group 1 students was MDT problem 4 (mean score .58), followed by MDT problem 6 (mean score .71). MDT problem 7 was ranked most difficult by 187 Group 1 students, followed by MDT problem 2 ranked most difficult by 96 students. MDT problems 3 and 5 were the least difficult for Group 1 students, and were identified by them as being the least difficult.

To measure the correlation between the ranking of MDT problem by mean scores and the ranking of MDT problems by difficulty as perceived by students, the non-parametric coefficient of correlation Kendall-tau, $\bar{\tau}$, was calculated. A value of 0 for $\bar{\tau}$ would mean independence for the two ordinal variables representing rankings. A value of 1 for $\bar{\tau}$ would correspond to perfect agreement.

By contrast, a value of -1 would correspond to perfect disagreement (Daniel, 1990, pp. 365-367). The hypothesis (one-sided) that the variables representing the two rankings are independent was tested:

H_0: The variables representing the ranking of mean scores for MDT problems, and the ranking of MDT problems by difficulty as perceived by students, are independent ($\tau = 0$).

H_1: $\tau > 0$.

The test statistic, $\bar{\tau}$, calculated using PASW18, was .43 and statistically non-significant for Group 0 students, and .52 and statistically significant for Group 1 students at $\alpha = .05$ (a priori established). Therefore, for Group 0 students, the hypothesis that the variables representing the two rankings are independent could not be rejected. Thus, there is no relationship between the variables representing the ranking of mean scores of MDT problems and the ranking of MDT problems by difficulty as perceived by students. However, for Group 1 students, the hypothesis that the variables representing the two rankings are independent (*p*-value = $.049 < \alpha$) could be rejected. Thus, there is a moderate agreement between the variables representing the ranking of mean scores of MDT problems and the ranking of MDT problems by difficulty as perceived by students.

The following summarizes the results for Research Question 2:

1. There is no agreement for Group 0 students and moderate agreement for Group 1 students between MDT performance and ranking of MDT problems by difficulty.

2. For both groups of students, MDT problems 3 and 5 (with the highest mean scores) were identified as the least difficult by students, whereas MDT problems 4, 6, and 7 (with the lowest mean scores) were not identified as most difficult for students.
3. All students could correctly identify the least difficult problems, but only Group 1 students could identify some of the difficult problems.

4.4 Research Question 3

Research Question 3: Does teacher ranking of problems by difficulty, after completing the MDT, correspond to student performance on the MDT?

All participating teachers (N=26) completed the Ranking Questionnaire for Teachers (RQT). The RQT is described in Chapter 3. Mean scores by MDT problem were calculated for all students, without distinguishing by group of students (Group 0 or Group 1). The lack of distinction was necessary because the teachers' Ranking Questionnaire responses did not discriminate between groups of students. The mean scores of MDT problems were ranked from lowest to highest, the most difficult MDT problem ranked the highest had the lowest mean. Frequencies obtained from RQT were ranked from highest to lowest, the MDT problem with the highest frequency being the one considered by teachers as the most difficult for their students. The ranking of student MDT mean scores was compared with the ranking obtained from RQT. The relationship between the variables representing the ranking of MDT problems by their mean scores and the

ranking by teachers of MDT problems by difficulty as perceived by their students, was assessed using the coefficient of correlation Kendall-tau (Daniel, 1990).

Table 41 presents the comparison of rankings by student performance on the MDT and rankings of problem difficulty by teachers. Roman numerals are used for ranking (e.g., "I" is the most difficult). The problem number is followed by parentheses with mean scores for the MDT problems in "Problem (Mean)" column, and with frequencies of the RQT responses in "Problem (Number of Teachers)" column.

Table 41

Student Performance on the MDT vs. Ranking of Problem Difficulty by Teachers

Ranking by MDT performance	Problem (Mean)	Ranking by RQT	Problem (Number of Teachers)
I	4 (.35)	I	7 (15)
II	6 (.51)	II	6 (6)
III	1 (.52)	III	5 (2)
IV	7 (.57)	IV*	1 (1)
V	2 (.64)	IV*	2 (1)
VI	3 (.78)	IV*	4 (1)
VII	5 (.81)	VII	3 (0)

Note. *Tied Rank (MDT problems 1, 2, and 4 were ranked most difficult by the same number of teachers; therefore they share the same rank, IV).

Like their students, when ranking MDT problems by difficulty, teachers also considered MDT problem 7 as one problem. Therefore, for ranking analyses purposes this problem was considered as one problem.

As can be seen in Table 41, the most difficult MDT problem for students was problem 4 (mean score of .35). Only one teacher identified MDT problem 4 as being most difficult for students. Teachers considered MDT problem 7 as most difficult for their students, which ranked fourth in the hierarchy of student performance on the MDT. Another difficult problem was MDT problem 6, ranked second in difficulty based on student performance on the MDT (mean score .51). MDT problem 6 was ranked second by teachers with respect to its difficulty. MDT problem 5, the least difficult problem (mean score .81) for students, was ranked third in terms of difficulty by teachers.

To assess the correlation between the ranking of MDT problem by mean scores and teachers' ranking of MDT problems by difficulty as perceived by students, the non-parametric coefficient of correlation Kendall-tau, $\bar{\tau}$, was calculated, and the hypothesis (one-sided) that the variables representing the two rankings are independent was tested:

H_0: The variables representing the ranking of mean scores for MDT problems, and teachers' ranking of MDT problems by difficulty as perceived by students are independent ($\tau = 0$).

H_1: $\tau > 0$.

The test statistic $\bar{\tau}$ was calculated using PASW18, and found to be .10, and statistically non-significant at $\alpha = .05$ (a priori established). Therefore, the

hypothesis that the variables representing the two rankings are independent could not be rejected. In conclusion, there is no relationship between the variables representing the ranking of mean scores of MDT problems and teachers' ranking of MDT problems by difficulty as perceived by students.

In summary, the results for Research Question 3 are the following:

1. There is no agreement between student performance on the MDT and teacher ranking of MDT problems by difficulty as perceived by their students.
2. Most teachers (15) considered MDT problem 7 as being most difficult for their students. However, when describing the nature of difficulty they did not notice that the most difficult part was 7b. They mentioned 7a as being difficult because the graph has three parts and three speeds for the car are required. They also mentioned 7c because students do not associate slope with rate of change. According to student performance, 7b was most difficult for students. Twice as many students correctly solved 7a and 7c, than 7b.
3. Teachers did not consider MDT problem 4 difficult for their students. According to student performance, problem 4 was surpassed in difficulty only by 7b. Both problems 4 and 7b require the calculation of the slope of a line when the graph is given.
4. Teachers did not consider MDT problem 1 to be difficult for their students. Six teachers considered MDT problem 6 difficult because their students did not learn about the *y*-intercept. Student performances on MDT problems 1 and 6 were about the same.

4.5 Research Question 4

Research Question 4: What is the nature of students' difficulties?

All participating students (N=1561) completed the MDT and Ranking Questionnaire for Students (RQS). The MDTs were scored using the Scoring Guide (SG), and students' responses from Ranking Questionnaires were coded using the Final Coding Categories (see Appendix B). To gain greater understanding of student difficulties with MDT aspects of linearity and linear functions, the researcher selected 40 students to be interviewed, two from each teacher who also agreed to be interviewed. In each pair of students, one student had an MDT total score above the group median, and one had an MDT total score below the group median. All interviews were conducted by the researcher. The Interviews data were reduced after the Exploratory Factor Analysis was performed, transcribed, coded using the Final Coding Categories (see Appendix B) and analyzed using the Linear Conceptual Field, a theoretical framework inspired by Vergnaud's conceptual field theory (1994, 2009).

4.5.1 Coding and Identifying the Nature of Student Difficulties from RQS.

The Ranking Questionnaires for Students (RQS) included students' descriptions of the nature of the difficulty for the MDT problem that posed the greatest difficulty for them. Their comments were coded and analyzed. Using the Preliminary Coding Categories (see Table 8, Chapter 3) the researcher and a second rater coded a sample of 50 Ranking Questionnaires and agreed to Final Coding Categories by reconsidering the definitions and boundaries of the

Preliminary Coding Categories. Table 42 presents descriptions of the Final Coding Categories for RQ written responses (see also Appendix B).

Table 42

Final Coding Categories for RQ Written Responses

Category	Description
H (Hard Problem)	A general statement about the difficulty of a problem, topic or domain. Sometimes a mathematical concept is identified as "hard" or "difficult." Other times a reference to unsatisfactory performance is made, but no specific details about the nature of the difficulty are given.
L (Never Learned/Taught)	A statement about the novelty of a problem, or that a mathematical concept or how to solve a certain problem was not learned or taught.
M (Mathematical Difficulty)	Nature of difficulty refers to lack of understanding of mathematical concepts or their representations, or difficulties with algorithms/procedures, computations, and symbolic manipulations.
R (Not Remembering)	A statement about forgetting or not remembering the mathematical concepts or how to solve the problem.
T (Thinking Time)	A reference to the amount of time, thinking, work, memory or any other resource necessary to solve a problem, or to a problem feature, as for example the number of subparts to be completed.
U (Not Understanding)	A general statement about not understanding a mathematical concept, topic, domain, how to solve a problem, or the vocabulary used, without offering any further details about the nature of the difficulty.

As can be seen in Table 42, there are six categories used for coding the participants' comments about the nature of difficulties. However, they are not the same as the preliminary six categories presented in Table 8 (Chapter 3). The major changes consisted of eliminating the preliminary categories *W* **(“**Holistic comment", e.g., "Algebra is hard"**)** and *O* **(“**Other", e.g., "The wording was not

clear”**)**; advancing new categories *L* (“Never Learned/Taught”) and *T* (“Thinking Time”); and reconsidering the boundaries of the categories *H* (“Hard Problem”), *M* (“Mathematical Difficulty”**)**, and *U* (“Not Understanding”). Comments in the category *W* referring to whole domains as being “hard” were moved into category *H*, and comments referring to ”not understanding” whole domains were moved into category *U*. Also, comments referring to not understanding the vocabulary were moved from category *O* to category *U*. Both categories *H* and *U* were redefined to include the comments from the old category *W*. The remaining comments falling initially in category *O*, referring to the amount of thinking time necessary to solve a problem, were moved into a new category, *T*, whose boundaries were enlarged by adding comments about the amount of work or memory needed to solve a problem. Finally, the category *M* was redefined by adding difficulties with algorithms/procedures (or steps of those), and with calculations or symbolic manipulations.

The coding for reduced data (MDT problems 2 and 7b were coded for Group 0 students, and MDT problems 1 and 4 were coded for Group 1 students) was conducted by the researcher and the second rater. An inter-rater reliability analysis using the Kappa statistic (Cohen, 1960) was performed to determine consistency among raters. The same assumptions were made for calculating the inter-rater agreement on RQ, as when calculating the inter-rater agreement on MDT scoring (Cohen, 1960):

i) The units of analysis representing participants’ answers on RQ are independent.

ii) The categories representing the nature of student difficulties are of nominal scale, independent, mutually exclusive and exhaustive.

iii) The raters judge independently, and are equally competent.

The inter-rater agreement for the two raters is presented in Table 43.

Table 43

Inter-Rater Agreement on RQ Written Responses

Group of Participants	Observed Kappa (*k*)	95% Confidence Interval	
		Lower Limit	Upper Limit
Students Group 0	.94	.91	.98
Students Group 1	.90	.81	.99

As shown in Table 43, the inter-rater agreement on coding of RQ student responses was calculated for each of the following groups of participating students: Group 0 students who were enrolled in mathematics courses at their grade level or below their grade level, and Group 1 students who were enrolled in advanced mathematics courses, that is, courses above their grade level in the mathematics curriculum sequence. The inter-agreement value for coding responses of both groups of students, Group 0 and Group 1, was very high, with $k = .94$ ($p < .05$), 95% CI (.91, .98), and $k = .90$ ($p < .05$), 95% CI (.81, .99), respectively.

Analyses of the nature of student difficulties with MDT problems were performed for the reduced data, i.e., difficulties with MDT problems 2 and 7b were analyzed for Group 0 students and the nature of difficulties with MDT

problems 1 and 4 were analyzed for Group 1 students. Tables 44 and 45 present Group 0 student difficulties with MDT problems 2 and 7b, and Group 1 student difficulties with MDT problems 1 and 4, respectively. Examples of students' comments, by category of difficulty, are included in the tables. Comments have the following format: "Student written comment [problem number]." For example, the comment "'I thought that one was the hardest because those types are just hard to me.'[2]", is a comment about MDT problem 2. In both tables, the column labeled "Frequencies by Problem" shows, for each problem, the number of students with comments in a category, followed by the corresponding percentage in parentheses. Before each table, the text of each problem is reproduced to aid the reader (see Figure 6).

Problem 2

$y = 2x - 3$

Create a table of values for the equation. Complete 5 rows of the table.

x	y
a.	
b.	
c.	
d.	
e.	

Problem 7 (All parts are presented, although the focus of analysis is on 7b.)

The graph below represents the distance that a car traveled after different numbers of hours.

(7a) *Which part of the graph (R, S, or T) represents the hours when the car moved the slowest?*

(7b) *What is the slope of part R of the graph?*

(7c) *What is the speed of the car in part R?*

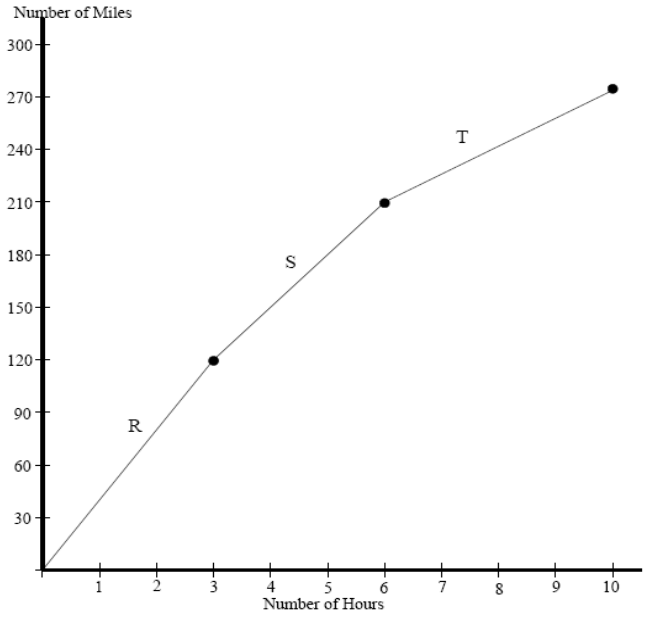


Figure 6. MDT Problems 2 and 7

Table 44

Group 0 Student Difficulties with MDT Problems 2 and 7b

Category	Examples	Frequencies by Problem	
		Problem 2 (N=158)	Problem 7b (N=136)
H (Hard Problem)	"I thought that one was the hardest because those types are just hard to me."[2] "This problem was difficult because the slope was difficult to find."[7b]	3(1.9%)	5(3.7%)
L (Never Learned/Taught)	"I didn't know from where to start. I didn't know the first step of the problem. I also haven't learned that problem yet."[2] "I hadn't seen anything like it. Don't get me wrong it was easy just new."[7]	5(3.2%)	1(0.7%)
M (Mathematical Difficulty)	"You had to figure out what numbers you had to put in because it doesn't tell you." [2] "There was no *x* value."[2] "I got confused on whether the slope was supposed to be 4/3 or 120/3." [7b]	19(12%)	33(24.3%)
R (Not Remembering)	"Forgot how to solve."[2] "I couldn't remember how to do the slope or how to find the slope of a line. Had a mind blank." [7b]	15(9.5%)	9(6.6%)
T (Thinking Time)	"At first I didn't know what it was asking so it took more thought then most questions."[2] "It took a long time for one problem."[7]	6(3.8%)	12(8.8%)
U (Not Understanding)	"Your wording was weird and I just didn't understand it!"[2] "I chose #7 because I do not understand slope very much at all."[7b]	70(44.3%)	51(37.5%)
No comment		40(25.3%)	25(18.4%)

As can be seen in Table 44, about two-thirds of the 158 students from Group 0 who identified MDT problem 2 as being most difficult made no comment about the nature of their difficulty, or stated that they did not understand the problem. Only 19 students who identified MDT problem 2 as most difficult described their difficulty as the identification of values for *x* (e.g., "I couldn't figure out what numbers to pick for *x*-value").

More than half of the 136 students from Group 0 who ranked MDT problem 7 as most difficult, either made no comment on the nature of the difficulty or commented, "I don't understand." Students' comments about the mathematical difficulty of MDT problem 7b identified the difficulty of figuring out which numbers to use in the formula to compute the slope (e.g., "In part *b* I didn't know whether to take from the values on the side or the units").

Note that about 16% of the 1040 students from Group 0 correctly solved MDT problem 4 requiring the calculation of the slope of a line graphed in a homogeneous system of coordinates, but failed to solve MDT problem 7b requiring calculation of the slope of a line graphed in a non-homogeneous system of coordinates. Also, it may be that problem context played a crucial role in students' success in solving the problems: 18% of Group 0 students who correctly solved MDT problem 7c that required finding the speed of the car (a real-life context) were not able to solve MDT problem 7b that required finding the slope of the line (a mathematical context).

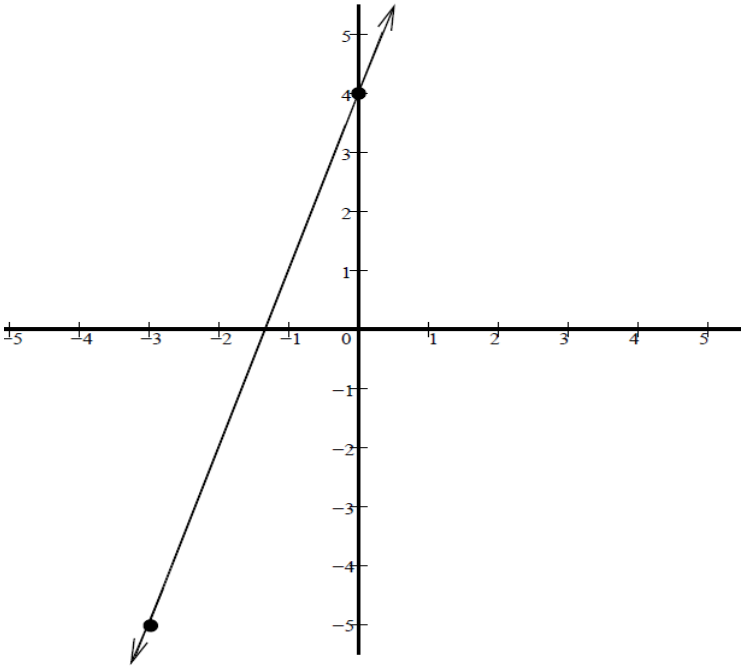
Table 45 presents difficulties with MDT problems 1 and 4 by category of difficulty for Group 1 students. For convenience, the texts of MDT problems 1 and 4 are presented (see Figure 7) before Table 45.

Problem 1

(a) Is the point with coordinates $(2,-8)$ *on the graph of the line* $y = 3x - 14$ *?*

(b) How did you decide?

Problem 4



In the graph above, what is the slope of the line? Show your work.

Figure 7. MDT Problems 1 and 4

Table 45

Group 1 Student Difficulties with MDT Problems 1 and 4

Category	Examples	Frequencies by Problem	
		Problem 1 (N=96)	Problem 4 (N=82)
H (Hard Problem)	"All the problems were easy." [1] "I was never good with slopes, or knowing how to find them, it's hard for me."[4]	1(1%)	2(2.4%)
L (Never Learned/Taught)	"I don't know how to find the slope. We're never taught it properly."[4]	0(0%)	1(1.2%)
M (Mathematical Difficulty)	"I have never heard the question asked that way. 'On the graph' could be anywhere."[1] "It wasn't a grid to help."[4] "I was not sure whether or not to put in decimal or keep it in fraction form."[4]	10(10.4%)	32(39%)
R (Not Remembering)	"I forgot how to do this type." [1] "I couldn't remember the formula for slope."[4]	22(22.4%)	18(22%)
T (Thinking Time) .	"It took me a long time to remember how to solve it."[1] "It was the problem I had to think on. All were easy just that problem made me think."[4]	15(15.6%)	9(11%)
U (Not Understanding)	"I didn't know it."[1] "No understandings of slope."[4]	38(39.6%)	13(15.9%)
No comment		10(10.4%)	7(8.5%)

In Table 45 it can be seen that almost half of the 96 students from Group 1 who identified MDT problem 1 as most difficult made no comment about the nature of their difficulty or made a comment about "Not Understanding." Only 10 students commented about the problem's "Mathematical Difficulty." Those students missed the Cartesian connection from point to line, had difficulty identifying

$(2,-8)$ as coordinates of a point (x_0, y_0), and checking whether the point $(2,-8)$ is situated on the line $y = 3x - 14$ (e.g., "I didn't understand at first what *x* and *y* were supposed to be, but I drew out a graph and then it just clicked in my head the *x* and *y* axis"). Interesting to note is that 14% of the 521 students from Group 1 who missed MDT problem 1 were able to correctly solve MDT problem 2, referring to the Cartesian connection from line to points.

Also of note is that about one-fourth of the 82 students from Group 1 who ranked MDT problem 4 as most difficult made no comment about the nature of the difficulty or made a comment about "Not Understanding." Two-fifths of the students who identified MDT problem 4 as most difficult commented on their mathematical or computational difficulty. Students identified the following obstacles concerning measurement: identifying the point where the measurement should start (e.g., "Didn't really give a place to start"), and assigning a magnitude in the absence of a grid line (e.g., "It wasn't a grid to help"). Students also had difficulty identifying the slope as "rise over run" (e.g., "I don't know what number goes on top or bottom"), assigning decimal numbers as magnitudes for rise or run, and then computing with those (e.g., "With the decimal it made it harder to turn into a slope").

Analyses of students' written comments on the nature of their difficulties included only 472 students, those Group 0 students who identified MDT problems 2 or 7 as being most difficult, and those Group 1 students who identified MDT problems 1 and 4 as most difficult. The following are the results of the analyses of student written comments:

1. More than one-half of students did not describe the nature of their difficulties.
2. About 28% of students described the nature of their difficulty by referring to the amount of time necessary to solve the problem, or difficulty remembering how to solve the problem.
3. Only 20% of students described their specific mathematical or computational difficulty.
4. For MDT problem 1 students had difficulty identifying $(2,-8)$ as coordinates of a point (x_0, y_0), and checking whether the point $(2,-8)$ is situated on the line $y = 3x - 14$.
5. For MDT problem 2, students had difficulty identifying *x*-values for the table of values.
6. For MDT problem 4, students had difficulty forming a ratio for slope.
7. For MDT problem 7b, students had difficulty determining the rise or change in *y*, since they had to decide between counting "tick marks" or taking the *y* values from the *Y*-axis into consideration.

4.5.2 Coding and Identifying Students' Mathematical Difficulties from Interviews. To gain greater understanding about students' conceptual knowledge and their mathematical difficulties with the Cartesian connection and slope, interviews were conducted with 40 students (28 students from Group 0 and 12 students from Group 1). Using the Preliminary Coding Categories from Tables 9-12 (see Chapter 3) the researcher and the second rater coded together five student

interviews and three teacher interviews, and agreed on definitions and boundaries for the Final Coding Categories. Agreement occurred when the same paragraph was coded by both raters using the same categories. Then using the Final Coding Categories the 40 student interviews were coded and analyzed (see Tables 46-49 and Appendix B). Tables 46 describes the Final Coding Categories for Conceptual Knowledge of Cartesian Connection (CKC).

Table 46

Final Coding Categories for Conceptual Knowledge of Cartesian Connection

CKC Categories	Description
CKC-D *From Point to Line*	A point (x_0, y_0) is on the graph of the line *l* if its coordinates satisfy the equation of *l*, $y = mx + b$, i.e., if the statement $y_0 = mx_0 + b$ is a true statement.
CKC-C *From Line to Points*	If a point (x_0, y_0) is on the graph of the line *l*, then its coordinates satisfy the equation of *l*, $y = mx + b$. There is an infinite number of points on a line, i.e., there is an infinite number of points (x_0, y_0) on the line *l* satisfying the equation of the line *l*, $y = mx + b$.

As can be seen in Table 46, one more category was added to the preliminary coding category from Table 9 (see Chapter 3) to account for the Cartesian connection from line to points in the situation from MDT problem 2.

Table 47 describes the Final Coding Categories for Conceptual Knowledge of Slope (CKS).

Table 47

Final Coding Categories for Conceptual Knowledge of Slope

CKS Categories	Description
CKS-P Physical Context	Slope describes the steepness of physical objects such as mountain roads, ski slopes, and wheelchair ramps. A statement about an entire class of objects sharing the property of having a slope or being steep, or an instantiation/particular example is given.
CKS-G Geometric-Analytical Context	Given two points (x_1, y_1) and (x_2, y_2), the slope of the line connecting the two points is $m = \frac{y_2 - y_1}{x_2 - x_1}$ (or "rise over run"). Homogeneous Systems of Coordinates: Slope is a geometric property of the line $y = mx + b$, as well as an analytical property of the function $y = mx + b$. Non-homogeneous Systems of Coordinates: Slope is an analytical property of the function $y = mx + b$, and not a geometric property of the line graphed.

In Table 47, there are only two final categories for conceptual knowledge of slope: 1) slope in a physical context and 2) slope in a geometric-analytical context. This is not surprising since Algebra I textbooks, used by participants in this study, introduce the concept of slope in physical and in geometric-analytical contexts (CKS-P) (Bennett et al., 2004; Larson et al., 2004; Rubenstein, et al., 1995). Also, the slope context for MDT problems 4 and 7b is geometric-analytical (CKS-G). In the Preliminary Coding Categories (see Chapter 3) the conceptual knowledge of slope referred to slope in other contexts, parametric, trigonometric, and covariational (CKS-A, CKS-T, and CKS-C). However, the inter-agreement analyses revealed that participants' answers referring to slope in other contexts could not account for knowledge of slope in those contexts (e.g., mentioning "*m*

in $y = mx + b$" was not considered evidence of conceiving of *m* as a parameter). Consequently, the final categories for conceptual knowledge of slope were limited to two contexts, physical and geometric-analytical.

Table 48 presents the Final Coding Categories for Student Difficulties with the Cartesian Connection (SDC) in the situations from MDT problems 1 and 2.

Table 48

Final Coding Categories for Student Difficulties with the Cartesian Connection

Categories	Subcategories	Description of Difficulty
SDC-D *From Point to Line* A point (x_0, y_0) is on the graph of the line *l* if its coordinates satisfy the equation of *l*, $y = mx + b$, i.e., if the statement $y_0 = mx_0 + b$ is a true statement.	*SDC-DP* Identifying the Point	Identifying $(2,-8)$ as (x_0, y_0) and replacing variables with their values in $y = 3x - 14$.
	SDC-DT Checking if True statement	Checking if the statement $-8 = 3(2) - 14$ (or other equivalent statement) is true.
SDS-C *From Line to Point(s)* If a point (x_0, y_0) is on the graph of the line *l*, then its coordinates satisfy the equation of *l*, $y = mx + b$. There is an infinite number of points on a line, i.e., there is an infinite number of points (x_0, y_0) on the line *l* satisfying the equation of the line *l*, $y = mx + b$.	*SDC-CX* Identifying *X*-values	Identifying values for *x*.
	SDC-CY Calculating *Y*-values	Using *x*-values to calculate the corresponding *y*-values, so that (x_0, y_0) satisfies $y = -2x + 3$.

As shown in Table 48, students' difficulties with the Cartesian connection occurred at each of the steps of the algorithm. Therefore, each of coding

subcategory represents one of these steps. In MDT problem 1, students had difficulty identifying the coordinates of a given point with values for the variables *x* and *y* from the equation of the line, substituting the coordinates of the point into the equation of the line, and checking if the equality statement is true. In MDT problem 2, students had difficulty identifying the *x*-values necessary to start completing a table of values for the function, and calculating the corresponding *y*-values.

Table 49 provides the Final Coding Categories for Student difficulties with Slope (SDS) by category and subcategory.

Table 49

Final Coding Categories for Student Difficulties with Slope

Categories	Subcategories	Description
SDS-R *Making and Calculating a Ratio* Making the rise (change in *y*), run (change in *x*), their ratio (slope), and calculating the ratio "rise over run"	*SDS-MH* Measurement in a Cartesian Homogeneous System of Coordinates	Identifying the rise and the run, and their magnitudes by measurement in a "slope triangle" (a triangle with two vertices being points on the line, (x_1, y_1) and (x_2, y_2) and the third vertex being either (x_1, y_2) or (x_2, y_1). Measurement refers to coordinating the origin point and the endpoint of measurement, the unit of measurement, and the sense of measurement.
	SDS-MN Measurement in a Cartesian Non-Homogeneous System of Coordinates	Identifying the unit of measurement by taking in consideration the values of the function represented by the line, and not the geometric representation of the line.
	SDS-CR Calculating a Ratio	Calculating the ratio rise/run

The categories identifying student difficulties with slope presented in Table 49, refer only to the geometric-analytical context proposed by MDT problems 4 and 7b.

An inter-rater agreement analysis using the Kappa (*k*) statistic (Cohen, 1960) was performed to determine the consistency among raters.

Inter-rater agreement analyses for Conceptual Knowledge (CKC, CKS) and Student Difficulties (SDC, SDS) were performed after the data were reduced, for the 40 student interviews, and was high, from $k = .80$ ($p < .05$), 95% CI (.63, .97) to $k = 1$.

4.5.2.1 Student Difficulties with the Cartesian Connection. Mathematical knowledge of the Cartesian connection can be stated as a mathematical theorem and its converse. When students described their solutions to MDT problems 1 and 2, they did not refer to the mathematical theorem, but rather identified their schemes of action for solving the problems. Those schemes of action were not always successful.

4.5.2.1.1 Student Difficulties with the Cartesian Connection in MDT Problem 1. Nine of the interviewed students from Group 1 solved MDT problem 1 using the following algorithm:

Step 1. Substitute the ordered pair $(2,-8)$ for (x_0, y_0) in $y = 3x - 14$; and

Step 2. Check whether the statement $-8 = 3(2) - 14$ is true.

The checking step was clearly stated before deciding whether the point was on the line (e.g., "…and -8 does equal -8, so it does lie on the line").

The following is a description of a student's successful solution of MDT problem 1 following the steps of the algorithm described above:

> I will substitute in the x value for 2 and the y value as -8. So my equation would now be $-8 = 3(2) - 14$. First I multiply 3 by 2 and that equals 6 and I subtract 14; $-8 = 6 - 14$; and that is -8, and $-8 = -8$. That's a check so I would say, yes, the point $(2,-8)$ is on the line $y = 3x - 14$.

Three students from Group 1 had difficulty with the Cartesian connection from point to line in MDT problem 1. One student plotted the point $(2,-8)$ and graphed the line $y = 3x - 14$ (see Figure 8). Since the point $(2,-8)$ was not on her line, the student concluded that the point could not be on the line $y = 3x - 14$, without questioning the accuracy of her graphing method.

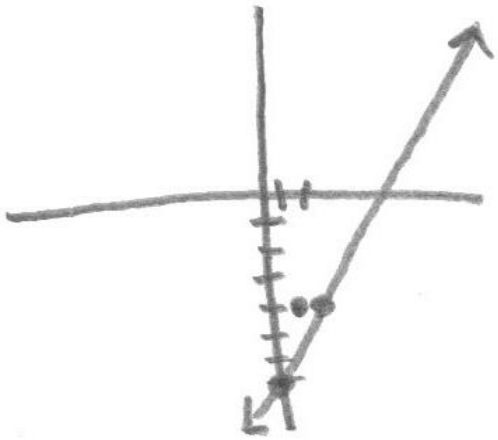


Figure 8. Example 1 of Student Answer to MDT Problem 1

Another student manipulated the equation $y = 3x - 14$ and correctly solved for x (see Figure 9). From his representation of the equation, he could not decide whether the point $(2,-8)$ is situated on the line.

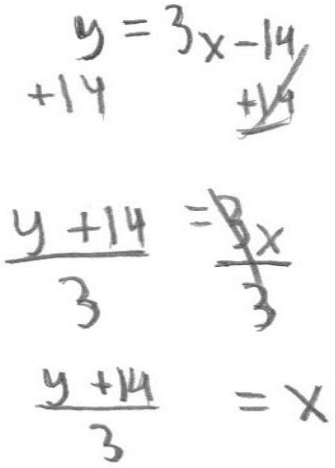


Figure 9. Example 2 of Student Answer to MDT Problem 1

None of the schemes tried by the third student were successful (see Figure 10).

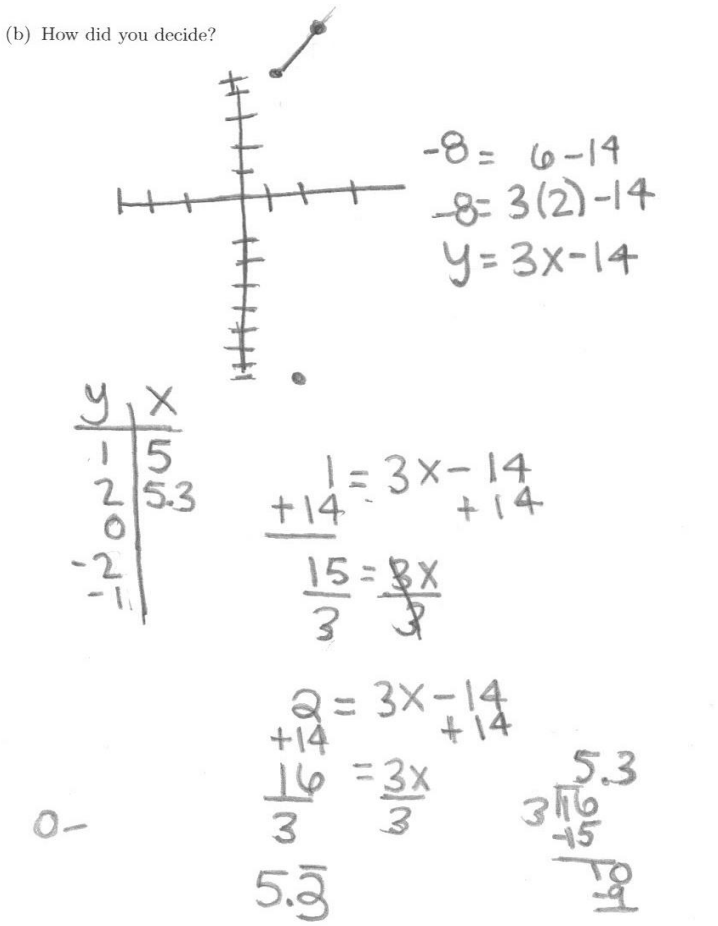


Figure 10. Example 3 of Student Answer to MDT Problem 1

As can be seen in Figure 10, the student identified $(2,-8)$ as coordinates (x_0, y_0) and replaced the variables *x*, *y* with those values in $y = 3x - 14$, but failed to check whether the statement $-8 = 6 - 14$ is true. She also tried making a table of values, graphing the line $y = 3x - 14$ using the table of values, and plotting the point $(2,-8)$. Her description follows:

I draw a graph first. I put in the numbers $y = 3x - 14; -8 = 3(2) - 14;$ $-8 = 6 - 14 \ldots$but I don't know what I'm doing. So I set up a table. Then, you plug all the numbers in and you draw a line, this is going up. So $(2,-8)$ it's not there.

4.5.2.1.2 Student Difficulties with the Cartesian Connection in MDT

Problem 2. All interviewed students from Group 0 (N=28), with the exception of two, successfully solved MDT problem 2 using the following algorithm:

Step1. Identify values for *x*; and

Step 2. Substitute the *x*-values in $y = -2x + 3$, and calculate the *y*-values.

One student explains how she solved MDT problem 2 (see Figure 11):

"I just picked random numbers for *x* and then I plugged the numbers I chose into the equation and got the *y* values."

	x	y
a.	-2	7
b.	-1	5
c.	0	3
d.	1	1
e.	2	-1

Figure 11. Example 1 of Student Answer to MDT Problem 2

Only one of the interviewed students remarked on the pattern "subtract 2 every time" and used it to complete the last row of his table (e.g., "…4 times -2 is -8

plus 3 is -5 and it's going by 2 so this is -7"). All other students calculated each of the *y*-values.

The absence of given *x*-values created an insurmountable obstacle for one student who failed to solve MDT problem 2:

> "Because it didn't have any numbers, patterns to start off with, then I wasn't able to really solve it."

Overcoming the obstacle would imply, albeit at the operational level, mathematical knowledge about the existence of an infinite number of points (x_0, y_0) with x_0 and y_0 real numbers, satisfying the equation of the line $y = -2x + 3$.

Students who successfully identified *x*-values, could still encounter difficulty calculating corresponding *y*-values. This would suggest knowledge that for a given x_0 there exists a unique *y*-value that satisfies the equation of the line $y = -2x + 3$. When such knowledge exists, the observed scheme of action would be substituting each of the *x*-values into $y = -2x + 3$ and calculating the corresponding *y*-values. However, in the absence of such knowledge, one student used the scheme of calculating the *y*-intercept of a line given the point $(x, 0)$, as shown in Figure 12. He gave the values -1, 0, 1, 2, 3 to *x*, considered $y = 0$, and calculated the values of the *y*-intercept *b* from $y = -2x + b$:

> I just put -1, 0, 1, 2, 3, and to find *y* just put it in, put the *x* value in the equation up here so *y* equals negative 2 times negative 1 plus *b*, and that's positive 2 so the *y* is negative 2 and then *y* equals negative two times 0

plus *b* and 0 and … then do the same thing for 1…Negative 2 times 1 plus *b*, negative 2, I add 2 here, would be positive 2 for *y*-intercept.

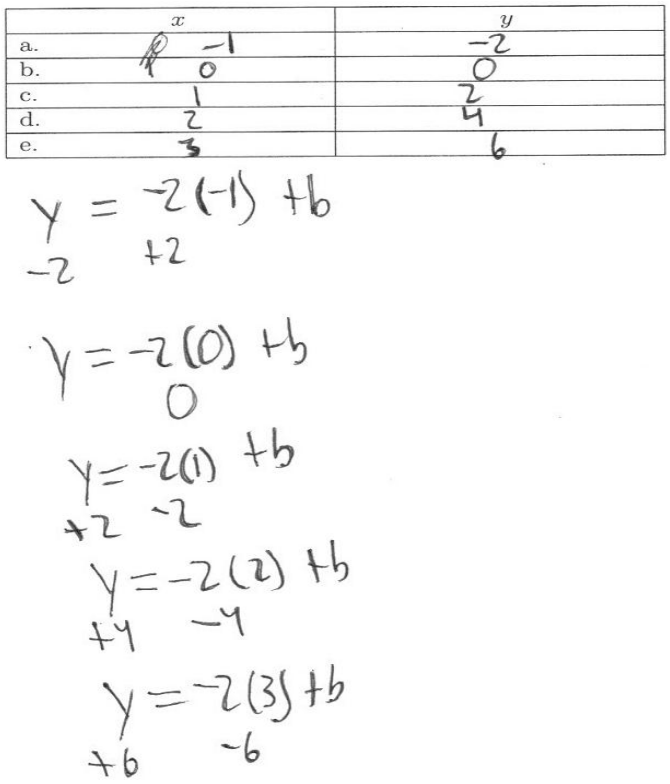


Figure 12. Example 2 of Student Answer to MDT Problem 2

Following is a summary of the results from interviews about difficulties with the Cartesian connection:

1. All students described their schemes of action without referring to the Cartesian connection as a formal mathematical theorem.
2. Students who missed the Cartesian connection from point to line in MDT problem 1, failed to identify the ordered pair $(2,-8)$ as (x_0, y_0) in

$y = 3x - 14$ or to check whether the statement $-8 = 3(2) - 14$ is true. They tried (unsuccessfully) schemes of action involving the construction of graphs or symbolic manipulations of the equation of the line.

3. Students who missed the Cartesian connection from line to points, illustrated in MDT problem 2, failed to identify *x*-values, or after replacing *x* with numerical values in $y = -2x + 3$, had difficulty calculating corresponding *y*-values.

4.5.2.2 Student Difficulties with the Concept of Slope in a Geometric-Analytical Context. Because both MDT problems 4 and 7b place slope in geometric–analytical contexts, the analysis of student difficulties focused on students' concepts of slope in a geometric-analytical context: given two points (x_1, y_1) and (x_2, y_2), the slope of the line connecting the two points is the "rise over run," or $m = \frac{y_2 - y_1}{x_2 - x_1}$. As mentioned in Chapter 3, students with a geometric perspective of slope (slope is an attribute of the line) used "slope triangles," counted the rise and the run, and calculated slope as "rise over run." Students with an analytical perspective of slope (slope is an attribute of the linear function) calculated the slope using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

4.5.2.2.1 Student Difficulties with Slope in MDT Problem 4. The algorithm used by students for calculating the slope in MDT problem 4 has two steps:

Step 1. Identify rise (change in *y*, or $y_2 - y_1$) and run (change in *x*, or $x_2 - x_1$) .

Step 2. Calculate "rise over run," or $\frac{y_2 - y_1}{x_2 - x_1}$.

The following are two examples of students' algorithms used to calculate the slope for MDT problem 4:

Student 1 (geometric perspective):

> To find the slope of a line I would do the rise over the run …I actually don't remember how to find the slope with just an equation or formula right now so I'm just going to go ahead and count… the rise is 1, 2, 3, 4, 5, 6, 7, 8, 9 and the run is 1, 2, 3, I believe. So it would be 9 over 3 because rise over run, or reduced, would be a slope of positive 3."

Student 2 (analytical perspective):

> You need to know the slope formula which is $\frac{y_2 - y_1}{x_2 - x_1}$ (*writing* $(0, 4)$, $(-3,-5)$, $\frac{-5-4}{-3-0} = \frac{-9}{-3} = 3$). I took the two points that they gave us and I put the *y*-s on top and I subtracted them and put the *x*-s on the bottom and I divided.

For Student 1, the geometric aspect of line prevailed and the student found the rise and run by counting the number of units. For Student 2, the analytical aspect of linear function prevailed and the student found the change in *y* and the change in *x* by calculating the differences $y_2 - y_1$ and $x_2 - x_1$, and their ratio.

Ten of the interviewed students from Group 1 correctly solved MDT problem 4. They identified the coordinates of two points from the graphed line and used

the formula $\frac{y_2 - y_1}{x_2 - x_1}$ to compute the slope. Only two of the interviewed students had difficulty finding the slope in MDT problem 4. Both students constructed triangles, sometimes referred to as "slope triangles" or just line segments, and used them to figure out (by counting) the rise and the run. One of the students who constructed a "slope triangle," drew two line segments, "rise" and "run" together with the line segment joining the two points of the given line, $(-3,-5)$ and $(0, 4)$ (see Figure 13). He correctly identified and calculated the rise and the run, but he failed to make the ratio "rise over run":

"I can't remember if I'm supposed to add 9 plus 3 or multiply or divide." The student identified the slope of the line as the ordered pair $(3, 9)$ since he could not make the ratio "rise over run."

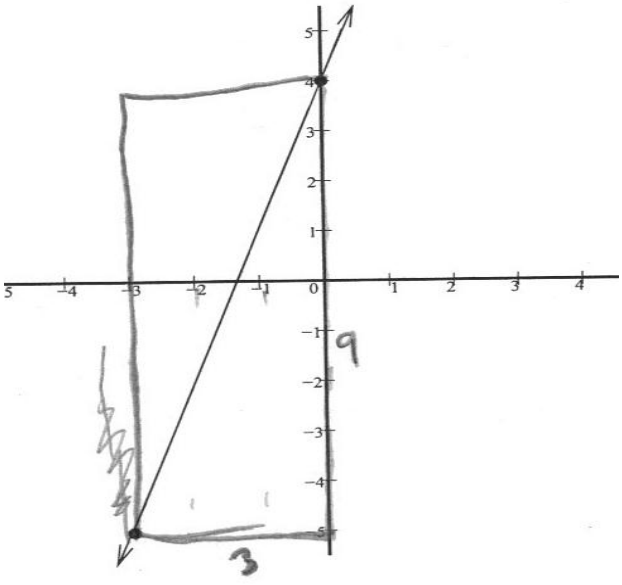


Figure 13. Example 1 of Student Answer to MDT Problem 4

The other student who had difficulties calculating the slope in MDT problem 4 drew a vertical line segment from $(-3,-5)$ to $(-3, 0)$ and identified it as the slope of the line (see Figure 14). The length of that line segment might be associated with the rise in a "slope triangle" with the vertices $(-3,-5)$, $(-3, 0)$, and (*x*-intercept, 0). The student associated the *x*-coordinate -3 of the endpoint $(-3, 0)$ situated on the *X*-axis with the measure for the rise; he "counted over 3 from 0" and took into consideration the negative sense of the *X*-axis:

> "I made a line up to the negative 3 and then counted over 3 from 0 (*indicating the vertical line segment he drew as slope).*"

The student did not mention any second quantity that might play the role of the run, and concluded that the slope of the line had the magnitude of -3.

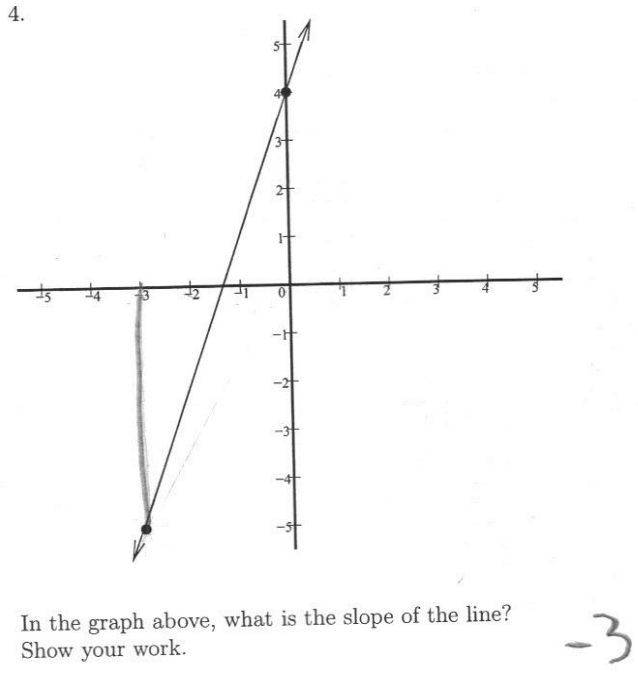


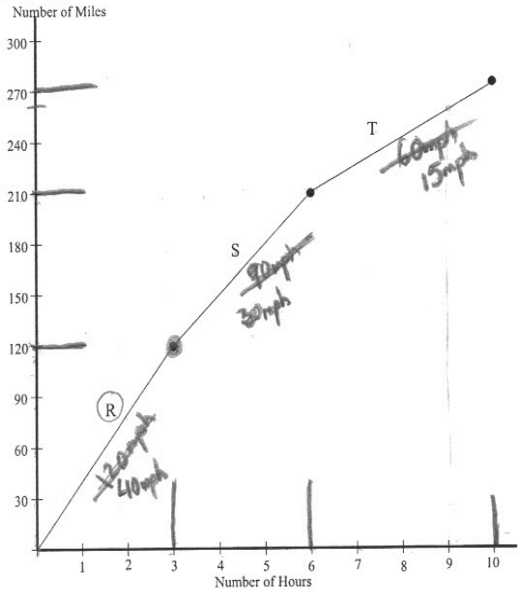
Figure 14. Example 2 of Student Answer to MDT Problem 4

In conclusion, the geometric perspective of slope prevailed in both schemes of action of the interviewed students who encountered difficulties with slope.

4.5.2.2.2 Student Difficulties with Slope in MDT Problem 7b. The importance of holding an analytical perspective of slope versus a geometric perspective of slope is further illustrated by students' difficulties with identifying the slope of a line graphed in a non-homogeneous system of coordinates as in MDT problem 7b. Nine of the 28 interviewed students from Group 0 held a geometric perspective of slope. They calculated the slope of part R of the graph as "rise over run," using the same unit of measurement on both coordinate axes, despite the fact that there was a different unit of measurement on the *Y*-axis. They found that the run, the line segment from the origin of the system of coordinates to (3, 0), had a length of 3 units on the *X*-axis, and the run, the line segment from the origin of the system of coordinates to (0, 120), had a length of 4 units on the *Y*-axis. They concluded that the slope of the line segment in MDT problem 7b is 4/3 (rather than 120/3). One student explained how he calculated the slope in MDT problem 7b (see Figure 15):

> "I go up 4 and over 3… slope is four thirds." [7b]

7. The graph below represents the distance that a car traveled after different numbers of hours.



(a) Which part of the graph(R, S, or T) represents the hours when the car moved the slowest?

T was when the car moved the slowest.

(b) What is the slope of part R of the graph?

the slope is 4/3

(c) What is the speed of the car in part R?

40mph

Figure 15. Example of Student Answer to MDT Problem 7

As can be seen in Figure 15, the student was able to figure out the speed of the car for R in MDT problem 7c, indicating no difficulty in reading or interpreting the graph. He no doubt holds a geometric perspective of slope, appropriate for

homogenous systems of coordinates, but not for non-homogeneous systems of coordinates where an analytical perspective would have been useful.

The following are results from interviews about student difficulties with slope:

1. The geometric perspective of slope prevailed in all schemes of action for the interviewed students who encountered difficulties with slope in MDT problems 4 and 7b.
2. For MDT problem 4, students attempted to construct "slope triangles," and failed to identify the rise, the run, or the ratio "rise over run."
3. In MDT problem 7b, students failed to take into consideration the values of the function for rise ($y_2 = 120$, $y_1 = 0$) when calculating $y_2 - y_1$, and instead counted the "tick marks" on the *Y*-axis (1, 2, 3). When the analytic perspective of slope prevailed, and students used $\frac{y_2 - y_1}{x_2 - x_1}$ to calculate the slope, the interviewed students did not encounter difficulties, except for minor numerical errors.

In summary, results from students' written comments and interviews showed that students had difficulty with the Cartesian connection and slope. Students missing the Cartesian connection from point to line had difficulty identifying a given ordered pair as Cartesian coordinates of a point, and checking whether the point is situated on a line, given its equation. Students missing the Cartesian connection from line to points had difficulty identifying *x*-values for a table of values and calculating the corresponding *y*-values, given the equation of the line. Students had difficulty measuring in a homogeneous system of coordinates and

forming a ratio for slope. Also, students had difficulty identifying the unit of measurement in a non-homogeneous system of coordinates. From the interviews, only those students with a geometric perspective of slope had difficulty figuring out the slope of a graphed line.

4.6 Research Question 5

Research Question 5: Do teachers' speculations about the nature of students' difficulties correspond to actual student difficulties as identified from student MDT performance and interviews?

All participating teachers (N=26) solved the MDT problems, completed the Ranking Questionnaire for Teachers (RQT) for which they ordered the MDT problems from most to least difficult from the perspective of their students, and commented on the nature of the difficulty for the two problems they considered to be most difficult for their students. For comparison purposes, teachers' written comments about student difficulties with the MDT problems were coded using the same Final Coding Categories as the ones used for coding students' written comments from the Ranking Questionnaires for Students (Table 42). The inter-rater agreement on coding of RQ written responses was computed for teacher response, and was high, $k = .89$ ($p < .05$), 95% CI (.74, 1).

Twenty teachers, seven from middle school and 13 from high school, were interviewed. Teachers' interviews were coded using the same Final Coding Categories as the ones used to code student Interviews (Tables 46-49). The inter-rater agreement was also high, from $k = .80$ ($p < .05$), 95% CI (.59, 1) to $k = 1$.

In the sections that follow, teachers' assessments of problem difficulty are compared with students' assessments of problem difficulty. Teachers' speculations about student difficulties with the Cartesian connection (MDT problems 1 and 2) and slope (MDT problems 4 and 7b), along with student difficulties identified from their performance on MDT (N=1561) and in interviews (N=40), are presented.

4.6.1 Teachers' Assessment of the Nature of Problem Difficulty. Table 50 provides examples of teachers' written comments on the Ranking Questionnaire for Teachers. The number in brackets identifies the MDT problem.

Table 50

Teacher Assessment of Problem Difficulty

Category	Examples	Number of Teachers
H (Hard Problem)		0
L (Never Learned/Taught)	"Haven't seen the material." [7]	10
M (Mathematical Difficulty)	"If the students plug the #s into the equation, they might know what to do next –simplify, then they may not know how to check if it is 'true' or not. If the students graphed the equation, a hand drawn graph may be hard to find the answer because hand drawn graphs can be hard to read." [1]	11
R (Not Remembering)	"Students forget how to find slope." [7]	1
T (Thinking Time)	"Lots of reading."[7]	3
U (Not Understanding)	"Most difficult: Mostly because of the wording. My students can solve it but the wording is not clear." [2]	1

As shown in Table 50, teachers did not consider any of the MDT problems as being "hard" for their students. Ten teachers (38%) identified the lack of familiarity with the problem's content or context (never seen, never taught/learned) as being the main factor for problem difficulty. Three teachers (9%) identified the amount of time necessary to solve a problem because of its features like, the length of the problem (e.g., MDT problem 7) or the format of the problem (e.g., MDT problem 1 is an extended–response type of problem requiring explanation of the solution procedure). Only one teacher (4%) mentioned that students have difficulty remembering the procedure for calculating the slope of a line. Also, only one teacher (4%) referred to student difficulty with the vocabulary used in a problem (MDT problem 2 requiring completion of a table of values). Although 11 teachers (42%) referred to the mathematical nature of problem's difficulty, none of those teachers described an actual student difficulty with slope. Only one teacher identified MDT problem 1 as being the most difficult for students missing the Cartesian connection, and only one teacher identified MDT problem 2 as being most difficult for students because of the vocabulary used.

4.6.2 Teachers' Comments on Student Difficulties with the Cartesian Connection.

4.6.2.1 MDT Problem 1: Cartesian Connection from Point to Line. On the MDT, 867 (55.5%) students did not correctly solve problem 1. Only one (3.8%)

of the 26 teachers ranked MDT problem 1 as most difficult. She described the mathematical difficulty students might encounter:

> If the students plug in the numbers into the equation, they might not know what to do next, simplify, then they may not know how to check if it is 'true' or not. If the students graphed the equation, a hand drawn graph, may be hard to find the answer because hand drawn graphs can be hard to read.

In other words, if a student attempted to use the Cartesian connection and replace (x_0, y_0) in the equation $y = 3x - 14$ with $(2,-8)$ then difficulty may arise at checking whether -8 equals $3(2) - 14$. If a student used the graphing strategy to solve MDT problem 1, and plotted the point and graphed the line, then difficulty may arise because hand drawn graphs may be inaccurate or difficult to interpret. Interviews revealed that students had difficulty identifying the ordered pair $(2,-8)$ as (x_0, y_0) in $y = 3x - 14$, checking whether the statement $-8 = 3(2) - 14$ is true, as well as deciding whether the point is on the graph after they plotted the point and graphed the line.

All interviewed teachers (N=20), except for the teacher mentioned above, were prompted by the researcher to comment on students' knowledge of the Cartesian connection from point to line and difficulties with MDT problem 1. Six teachers described their knowledge of the Cartesian connection from point to line as both a "theorem in action" (Vergnaud, 2009) and a solution algorithm:

> If the coordinates $(2,-8)$ are in the graph of that line, then they should fulfill that equation, so if you put them in, it should balance it, it should be equal on both sides.

Fourteen (70%) teachers described the Cartesian connection from point to line only procedurally, that is, as an algorithm for solving the problem:

> We can test it to see if it's a solution, by replacing the *x* and the *y* coordinates and if we simplify and it makes a true sentence it is a solution to the equation of two variables because it makes the sentence true, right?

Thirteen (65%) teachers identified the Cartesian connection as an obstacle for their students who could not solve MDT problem1, or tried to solve it using a graphing approach:

> Two is *x* and -8 is *y*; putting those coordinates in for *x* and *y* and making sure that it's a true statement. If they don't have, can make that connection with the point, they tried to graph it.

Three (15%) teachers identified student difficulties with MDT problem 1 as switching the *x*-value with the *y*-value in $(2,-8)$ when substituting those values into the equation of the line, $y = 3x - 14$:

> They have a problem of substituting, basically, and knowing an ordered pair. Knowing that *x* comes first, *y* comes second and knowing that they are substituting in these values into this equation.

Although most interviewed teachers, when prompted by the researcher, identified lack of understanding of the Cartesian connection as the major obstacle in MDT problem 1 for their students, and referred to the Cartesian connection

from point to line as an algorithm to solve the problem, they failed to suggest that student difficulties could arise at each of the steps of the algorithm.

4.6.2.2 MDT Problem 2: Cartesian Connection from Line to Points. On the MDT, 685 (43.9%) students did not correctly solve problem 2. Only one (3.8%) of the 26 teachers ranked MDT problem 2 as most difficult "mostly because of the wording, it is not clear."

Interviews revealed that students had difficulty identifying *x*-values to enter into the table, as well as substituting *x* values into the equation $y = -2x + 3$ in order to calculate corresponding values of *y*.

When prompted by the researcher, six (30%) of the interviewed teachers identified the mathematical difficulty as the absence of any given *x*-values to help students get started with the completion of the table:

> It's sort of open-ended in that they can fill in any value for *x* and finding *y* is going to be difficult, filling out that table, just because of how open-ended it was. If the problem included values for *x*, it might be a little bit less difficult for them.

Two (10%) of the interviewed teachers referred to the novelty of the format of the problem as a potential source of difficulty. According to them, the *x*-values were always given to their students, and the format of the table was different from the one in MDT problem 2.

Teacher 1:

> I always tell my students to create a table where we're going to be using the values of -2 to $+2$. I say that all the time. I let them know what points I want them to use.

Teacher 2:

> Our district testing, our state testing, and our books all do the tables the same way. I can see them looking at this table and going 'What?' because they're used to having to put the equation in the middle, plug it in, and then solving it. So, just the format, I can see them being confused by the format. Sometimes if they don't recognize the format, they just shut down.

None of the teachers commented on the difficulty calculating corresponding *y*-values once *x*-values are identified, although this type of difficulty arose for two of the interviewed students. Two (10%) of the interviewed teachers, when prompted by the researcher, commented about the necessity for students to conceive the existence of an infinite number of points (x_0, y_0) composing the line $y = -2x + 3$, and therefore satisfying its equation:

> They have to understand that this equation describes a relationship of all these points together and so if they don't understand that, they wouldn't know what to do. I'll show a line, but what I'll do is I'll go in and I'll just put points, I'll take my marker and just make dots all the way along it, and I'll say: "See all these points here? All these points make this equation true. So, any of these points are values of *x* and *y* that make it true."

As with problem 1, most teachers referred to the Cartesian connection from line to points illustrated in problem 2 as an algorithm to solve the problem. However, they failed to point out difficulties their students may encounter at each step of the algorithm.

4.6.3 Teachers' Comments on Student Difficulties with Slope in a Geometric-Analytical Context

4.6.3.1 MDT Problem 4: Slope in Homogeneous Systems of Coordinates. On the MDT, 1009 (71.4%) students did not correctly solve problem 4. Only one (3.8%) of the 26 teachers ranked MDT problem 4 as most difficult because "no points [were] labeled."

Interviews revealed that only students with a geometric perspective of slope encountered difficulties with figuring out the slope. Those students attempted to construct "slope triangles" and failed to identify the rise, the run, or the ratio "rise over run."

When prompted by the researcher, 18 (90%) of the interviewed teachers described their students' knowledge of slope in a geometric-analytical context:

> Well, in this case, there's various ways. You can pick any two points and do the change in *y* over the change in *x*. And then of course there's always the rise over the run thing, so you basically you use just a method of counting or if you actually use that formula. I think for most students […] making the segment a rate-triangle format and doing the rise over the run hopefully is a little bit easier for them.

Student interviews also revealed measurement difficulties when identifying the "run" for calculating the slope. This difficulty occurred when students chose points for the "run" that were not integers. Instead, they approximated the *x*-intercept of the line. When prompted by the researcher, four (20%) of the interviewed teachers commented on this type of difficulty:

> If we refer to the point where this line from problem 4 intersects *X*-axis they could give you many different values. They could give you 0, they would tell you that it's (0, 1.5), they'd tell you that it's (0, 1) because it's close enough to… they'll think that it's close enough, then they'll figure $(0,-1)$ because it's close enough to -1.

Two (10%) of the interviewed teachers suggested that the lack of gridlines or labels for the points in MDT problem 4 may have been the cause of students' measurement difficulties. However, student interviews did not reveal this type of difficulty. Only one (5%) of the interviewed teachers identified choosing or replacing values in the formula for slope as a mathematical difficulty in MDT problem 4 for students:

> Also another error that they make is they flip the *x* and *y* coordinates … they'll write down the coordinate right but when they go to plug it into their $\frac{y_2 - y_1}{x_2 - x_1}$, they'll use the *x* and *y* of one coordinate, of the same coordinate, and put that on top and they put the *x* and *y* on the bottom. I can write that out for you so that you kind of get an idea. They'll have their (x_1, y_1) and their (x_2, y_2) and they'll…for their slope, they'll do

$y_2 - x_2$ instead. They'll use just one point and be like, O.K., I'm starting with this point and that's my second point so I'm going to do that one on top and they do my $y_1 - x_1$, and so they'll get the formula confused, they'll… So they mistake which one is *x* and which one is *y*.

All interviewed students who employed the formula $\frac{y_2 - y_1}{x_2 - x_1}$ to compute the slope, used it correctly.

In general, teachers underestimated the difficulty of identifying the slope in problem 4. When prompted by the researcher to comment on the nature of their students' difficulties with slope, teachers did not mention any of the difficulties associated with the construction of a "slope triangle."

4.6.3.2 MDT Problem 7b: Slope in Non-Homogeneous Systems of Coordinates.

On the MDT, 1115 (71.4%) students did not correctly solve problem 7b. Fifteen (55.6%) teachers identified problem 7 (7a, 7b, and 7c) as being most difficult for their students. Twelve (44.4%) teachers described the major difficulty as both misreading and interpreting the graph, particularly interpreting the graph formed by the three line segments (7a), and the computing of the speed of the car (7c). Only when prompted by the researcher did 12 (60%) of the interviewed teachers comment on the nature of the difficulty with problem 7b as calculating the slope of 4/3:

They're counting the rise over the run. So they're going over, they know how many spaces they're going over, but they're counting the spaces,

> they're not counting the quantity. So they're going over 3 like they should, they know that they're going over 4, but they're not associating 4 with this value.

Only one of the interviewed teachers (5%), and only when prompted, commented on the necessity of explaining how to measure in a non-homogeneous system of coordinates:

> I would explain it to them that that tick mark isn't just one. What does it say next to it? Is it 30? So if that one tick mark is 30 is it really that I'm going up 1 or how many am I really going up? So it's a matter of paying attention to what the intervals are on the graph.

Although none of the interviewed teachers identified measuring in non-homogenous systems of coordinates as the main mathematical difficulty for their students, when prompted by the researcher, they could describe the nature of the difficulty that led students to answer 4/3 instead of 120/3 to problem7b.

Analyses of teachers' written comments and interviews about the nature of their students' difficulties, showed that:

1. Speculations about the nature of student difficulties did not correspond to actual student difficulties, as identified from student performance on the MDT and student Interviews.
2. Only 11 (42%) of the 26 teachers who completed RQT described a mathematical difficulty. Only six of those 11 teachers described a mathematical difficulty that was actually encountered by students, namely

missing the Cartesian connection (MDT problem 1) and considering slope as rate of change (MDT problem 7b).

3. During the interviews, only four (20%) teachers described a mathematical difficulty that corresponded to actual student mathematical difficulties. When prompted by the researcher to comment on a specific type of difficulty, the percentage of interviewed teachers that described an actual mathematical difficulty encountered by students improved. With prompting, 13 (65%) teachers described missing the Cartesian connection in MDT problem 1, six (30%) teachers described the difficulty of finding *x*-values for the table in MDT problem 2, four (20%) teachers referred to measurement difficulties when estimating the *x*-intercept in MDT problem 4, and four (20%) teachers described the difficulty measuring the run or the change in *y* in MDT problem 7b.

Chapter Summary. In this chapter the data, analyses of data and results of the analyses were presented by research question. The main results are the following:

i. There was little variation in student performance on the MDT from Grade 8 through Grade 10. Student performance on the MDT increased with more advanced courses, mainly due to Group 1 performance. Group 1 students outperformed Group 0 students.

ii. All students knew which of the problems were correctly solved, and therefore they could correctly identify the problems with the highest mean scores as being the least difficult problems. Only Group 1 students could identify some of the problems with lower mean scores as being difficult

for them. Teachers could not identify the MDT problems that were most difficult for their students.

iii. The most difficult problems for participating students were those requiring identification of the slope of a line from its graph, MDT problems 4 and 7b. That difficulty persisted across grades and mathematics courses. The geometric aspect of slope prevailed in all schemes of action for the interviewed students who encountered difficulties with those problems. For MDT problem 4, students attempted to construct "slope triangles," and failed to identify the rise, the run, or the ratio "rise over run." In MDT problem 7b, students failed to take into consideration the values of the function for rise and instead, counted the "tick marks" on the Y-axis. When the analytic aspect of slope prevailed and students used $\frac{y_2 - y_1}{x_2 - x_1}$ to calculate the slope, the interviewed students did not encounter difficulties, except for minor numerical errors.

iv. Teachers' comments about the nature of their students' difficulties did not agree with student difficulties as identified from student performance on the MDT and Interviews. The agreement remained low, even when teachers were prompted by the researcher to comment on specific student difficulties (e.g., students' difficulty with slope in MDT problem 7b).

In Chapter 5, results of this study are discussed in relation to other research on students' understanding of aspects of linearity and teachers' knowledge of student difficulties. Conclusions that can be drawn from the results of the data analyses,

along with limitations for generalizing results and recommendations for further research, are presented.

Chapter 5

DISCUSSIONS AND CONCLUSIONS

5.1 Introduction

To meet the national need for more mathematicians, scientists, engineers and highly skilled workers, high schools are now requiring completion of the Algebra I, Geometry, and Algebra II sequence for entrance into college and the workplace. Of those courses, Algebra I is singled out as fundamental preparation for the study of higher-level mathematics and the sciences (Gamoran & Hannigan, 2000; Katz, 2007; Matthews & Farmer, 2008; Smith, 1996; Speilhagen, 2006). Among the key concepts developed in Algebra I are linear equations and functions (National Mathematics Advisory Panel, 2008). During the last two decades, student difficulties with aspects of linearity and linear functions have been studied by many researchers with various populations of students (Ben-Chaim et al., 1998; Cheng, 2010; Davis, 2007; Ellis, 2009; Hines, 2002; Knuth, 2000; Lamon, 1993; Lobato & Siebert, 2002; Moschkovich et al., 1993; Newburgh, 2001; Orton, 1983; Schoenfeld et al., 1993; Smith et al., 1989; Stump, 2001a; Zaslavsky et al., 2002). Despite their efforts to gain insight into student difficulties and assist mathematics educators in implementing curricular and instructional changes, student difficulties with linearity and linear functions persist (American Diploma Project, 2010).

In prior research studies, with few exceptions (Even & Tirosh, 1995; You, 2006), student understanding and difficulties were assessed without input, in the form of descriptions, from students or their teachers (Cheng, 2010; Davis, 2007;

Knuth, 2000; Orton, 1983; Stump, 2001a; Zaslavsky et al., 2002). By contrast, the present study not only tested students on concepts of linearity, but also investigated their assessments of their own difficulties and their mathematics teachers' understanding of the nature of their students' difficulties.

In this chapter, results of data analyses are summarized, and discussions of those results with respect to previous research findings are presented by research question (Section 5.2). Limitations of this study (Section 5.3) and recommendations for future research (Section 5.4) conclude the chapter.

5.2 Results and Discussion

The following questions were addressed in this study:

1. Does student performance on the Mini-Diagnostic Test (MDT) vary by:

    a. Grade level?

    b. Current mathematics course?

    c. Problem?

2. Does student ranking of problems by difficulty, after completing the MDT, correspond to their performance on the MDT?

3. Does teacher ranking of problems by difficulty, after completing the MDT, correspond to student performance on the MDT?

4. What is the nature of students' difficulties?

5. Do teachers' speculations about the nature of students' difficulties correspond to actual student difficulties as identified from student performance on the MDT and interviews?

5.2.1 Research Question 1 Results and Discussion.

Research Question 1: Does student performance on the Mini-Diagnostic Test (MDT) vary by:

a. Grade level?

b. Current mathematics course?

c. Problem?

To answer Research Question 1, the Information Questionnaire for Students (IQS) and the Mini-Diagnostic Test (MDT) were administered to all students. Data from IQS (grade level, current mathematics course) and MDT scores were analyzed.

Results for Question 1 revealed little variation in student performance on the MDT across grade levels. Regardless of grade level, on average, students correctly solved five of the nine MDT problems. However, analyses of the MDT score distributions revealed greater numbers of higher scores with increasing grade level. The presence of higher scores at higher grade levels may be due to increased exposure to the concepts by virtue of completion of more advanced mathematics courses.

With respect to variations in performance as related to current mathematics course, results showed that Group 1 students, enrolled in mathematics courses

above their grade level in the mathematical instructional sequence, outperformed Group 0 students, enrolled in courses at or below their grade level in the sequence. Performance gains on the MDT across courses were due mainly to Group 1 students' score contributions. These results might be explained by the relationship between the type of learners participating in this study and their current mathematics courses. Group 0 students, identified as low-achieving or average-achieving and on remediation or median tracks, were enrolled in lower division mathematics courses (e.g., Pre-Algebra, Algebra I) . By contrast, Group 1 students, identified as high-achieving and on the college-track, were enrolled in more advanced courses, such as Geometry or Algebra II. These performance results, as well as their justifications, are supported by Gamoran and Hannigan's research (2000) and by the American Diploma Project (2010). During a longitudinal study, Gamoran and Hannigan (2000) adminstered achievement tests to students in Grades 8 and 10, and found that those who completed Algebra I earlier in the instructional sequence outperformed students who took Algebra I later or did not take Algebra I . The American Diploma Project (ADP) reported that Algebra I student performance on end-of-course examinations decreased by grade level. That is, students who completed Algebra I in earlier grades outperformed students who completed Algebra I later in their education. Although those two research studies had results in concert with the results of the present study, their assessments differed. In both Gamoran and Hannigan's and ADP's studies, assessments were more comprehensive (i.e., included topics in addition to linearity) and were tied to course completion (i.e., administered at the end of an

instructional sequence). By contrast, the present study measured student performance on only aspects of linearity and linear functions, was administered during the instructional period and not at the end of a course, and was not restricted to Algebra I students.

With regard to variations in performance based on problem type, students had greatest difficulty with MDT problems focusing on slope (MDT problems 4 and 7b, identifying the slope given the graph of the line), and the Cartesian connection (MDT problems 1 and 2, Cartesian connection from point to line, and line to points, respectively). Student difficulties with slope and the Cartesian connection persisted across grades, mathematics courses, and types of learners (Group 0 and Group 1). Student performance on MDT problems 1, 2, 4, and 7b increased with enrollment in more advanced mathematics courses, like Geometry and Algebra II. One possible explanation for higher MDT scores for Geometry and Algebra II students, may be due to their completion of more mathematics courses at more advanced levels. Findings of the present study are in concert with the three-country study using the MDT conducted by Greenes et al. (2007a) that found that U.S., Israeli and Korean students had difficulties with slope (MDT problems 4 and 7b). Furthermore, U.S. students had greater difficulty with the Cartesian connection (MDT problems 1 and 2) than did the Korean and Israeli students. Those differences in performance, even at the same grade level, suggest that curricular and instructional practices influenced student performance, and that U.S. curricula, versus curricula and pedagogical practices in Korea and Israel,

may place less emphasis on making connections among various representations of functions.

In conclusion, student performance on the MDT varied little across grades, but increased with more advanced mathematics courses, mainly due to Group 1 students who were enrolled in courses above their grade level expectations in the instructional sequence. These findings are consistent with the results of the studies conducted by Gamoran and Hannigan (2000), and the American Diploma Project (2010). The most difficult problem across grades and mathematics courses was the problem of identifying the slope of a line, given the graph of the line (MDT problems 4 and 7b). Results from previous studies with similar or same problems, confirmed the persisting difficulty of mastering the concept of slope in geometric-analytical contexts (Greenes et al., 2007a; Knuth, 2000; Orton, 1983; Stump, 2001a; Teuscher & Reys, 2010; Zaslavsky et al., 2002).

5.2.2 Research Question 2 Results and Discussion.

Research Question 2: Does student ranking of problems by difficulty, after completing the MDT, correspond to their performance on the MDT?

To answer Research Question 2, the Mini-Diagnostic Test (MDT) and Ranking Questionnaire for Students (RQS) were administered to all students. The MDT mean score for each of the MDT problems was used to obtain the empirical ranking of relative problem difficulty. Frequencies of student rankings of problem difficulty from the RQS were used to obtain student subjective ranking of relative problem difficulty. The empirical ranking based on student

performance on the MDT and the subjective ranking based on student perception of difficulty were compared.

Results for Question 2 showed no statistically significant agreement for Group 0 students and moderate agreement for Group 1 students between their empirical rankings of difficulty based on MDT performance and their rankings of MDT problems by difficulty. A relative comparison of rankings showed that all students could correctly identify the least difficult problems, but only Group 1 students could identify some of the difficult problems. In other words, all students knew what they knew well, but only the high-achieving students somewhat knew what they did not know on MDT. Overall, for both groups of students, MDT problems 3 and 5, with the highest mean scores, were identified as the least difficult, whereas MDT problems 4, 6 and 7 with the lowest mean scores, were not identified as most difficult.

There is a paucity of studies that examined students' perceptions of problem difficulties as compared to their actual performance. A study conducted with Australian undergraduate students enrolled in Biochemistry and Physics courses (Lingard et al., 2009) found a similar result of moderate agreement between the ranking based on student performance and the ranking based on student perception of difficulty. Another study in Hong Kong found that Grade 9 students' perceptions of problem difficulty correlated highly with their performance on an algebra test on logarithms (Lee & Heyworth, 1996). These findings suggest that student cultural background, curricular experiences, type of

learner, and the problem content and context may play important roles not only in student performance, but also in students' perceptions of problem difficulty.

5.2.3 Research Question 3 Results and Discussion.

Research Question 3: Does teacher ranking of problems by difficulty, after completing the MDT, correspond to student performance on the MDT?

To answer Research Question 3, the Mini-Diagnostic Test (MDT) was administered to all participating students and teachers, and the Ranking Questionnaire for Teachers (RQT) was administered to all participating teachers. Student performance, given by the MDT mean score for each of the MDT problems, was used to obtain the ranking of problem difficulty. Teacher assessment of problem difficulty was given by the frequencies of teacher rankings of problem difficulty from the RQT. Of note, teachers' rankings of problem difficulty referred to problem difficulty as perceived by their students. The ranking based on student performance and the ranking based on teacher assessment of problem difficulty were compared.

The comparison between teachers' ranking of problem difficulty and their students' performance showed that teachers were not aware of their students' difficulties with slope (MDT problems 4 and 7b) and the Cartesian connection (MDT problem1). For example, none of the teachers identified MDT problem 4 as being the most difficult for their students.

Results showed no significant agreement between student performance on the MDT and teacher ranking of MDT problems by difficulty. By contrast, with the

exception of the study by Lingard et al. (2009), other studies showed low or moderate agreement (Koedinger & Nathan, 2004; Lee & Heyworth, 1996, 2000; Lorge & Kruglov, 1953; Nathan & Koedinger, 2000; Zhou, 2009), or high agreement (Gire & Rebello, 2010). However, even in the case of the study by Lingard et al. (2009), only the level of difficulty of the problem on a scale with three values was identified. In all of these studies, since the populations, contents, and methods of administration differed, comparisons are difficult to make.

5.2.4 Research Question 4 Results and Discussion.

Research Question 4: What is the nature of students' difficulties?

To answer Research Question 4, the Ranking Questionnaire for Students (RQS) was administered to all participating students, and interviews were conducted with 40 students (see Chapter 3). Students' written comments from the RQS about the nature of their difficulties, and students' interviews were analyzed. The focus of analysis was on the mathematical nature of difficulties encountered by the students (N=472) who identified problems referring to slope or the Cartesian connection as being difficult for them.

In their written comments, about 51% of the 472 students did not describe their difficulties, and 28% of students referred to the resources necessary to solve a problem (e.g., amount of time, memory). The positive correlation between student assessment of problem difficulty and amount of time needed to solve the problem also was identified by Lingard et al. (2009). Taking into consideration the association between problem characteristics and the number of solution steps,

the findings from the present study are in accord with Lee and Heyworth's (1996; 2000) findings about student perceived problem difficulty that could be predicted by, among other factors, the number of algorithmic steps and the number of computational operations necessary to solve a problem.

Only 20% of the 472 students in the present study made comments about their specific mathematical difficulty. Analyses of the nature of students' written comments and interviews revealed that students had difficulty with the Cartesian connection and slope. Difficulties with the Cartesian connection included identifying the Cartesian coordinates of a point and checking whether the point is situated on a line, given the coordinates of the point and the equation of the line (MDT problem 1); and identifying *x*-values for a table of values and calculating corresponding *y*-values, given the equation of the line (MDT problem 2). Difficulty recognizing points (x_0, y_0) as solutions of the linear equation $y = mx + b$ or $ax + by = c$ is confirmed by studies conducted by Greenes et al. (2007a), Knuth (2000), Moschkovich et al. (1993), and Schoenfeld et al. (1993).

Student difficulties with determining the slope of a line, given the graph of the line (MDT problems 4 and 7b, included: 1) measuring (e.g., not connecting the rise and run of the slope of the line with the corresponding oriented segments, positive or negative); 2) forming a ratio for slope (e.g., attending only to the rise, change in *y*); and 3) identifying the unit of measure in a non-homogeneous system of coordinates (e.g., counting the "tick marks" on the *Y*-axis and disregarding the scale). Difficulty connecting $x_2 - x_1$ (change in *x* or run) or $y_2 - y_1$ (change in *y* or rise) with their graphic representations and with their corresponding positive

or negative sign, is also reported by Greenes et al. (2007a), Moschkovich et al. (1993), Schoenfeld et al. (1993), and Teuscher and Reys (2010). Results of the present study as well as results from previous studies (Cheng, 2010; Kaput & Maxwell-West, 1994; Lamon, 1993; Lobato & Siebert, 2002) suggest that identifying the slope of a line graphed in a system of Cartesian coordinates is a more difficult task than identifying the rate of change of a linear function when two instances of the varying quantities, (x_1, y_1) and (x_2, y_2), are given. One explanation may be that the graphing context for slope implies knowledge of mathematical conventions (e.g., if the oriented segment representing the rise is toward the positive sense of the *Y*-axis, its magnitude has a positive sign), considered by Schoenfeld et al. (1993) as part of the Cartesian connection. Thus, difficulty in identifying the slope of a graphed line lies with the connection with the graphic representation. This explanation is supported in the present study by findings from students' Interviews; only those students with a geometric perspective of slope, (i.e., students who calculated slope as "rise over run") had difficulty figuring out the slope of the graphed line in MDT problems 4 and 7b. Another explanation may be that the complexity of the problem increased for those students with a geometric perspective of slope (Kaput & Maxwell-West, 1994; Lamon, 1993; Lobato & Siebert, 2002). Indeed, calculating the slope of a graphed line as "rise over run," implies knowledge of proportionality in the geometric context of similar triangles, or at minimum, coordination while identifying the rise and the run from a "slope triangle." By contrast, students with an analytical perspective of slope, who calculated slope as "change in *y* over

change in *x*," or used the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, had less difficulty since the connection with the graphic representation was reduced to identifying the coordinates of the points, (x_1, y_1) and (x_2, y_2). Nevertheless, studies conducted by Lobato and Siebert (2002), Orton (1983), and Stump (2001a) suggest that the concept of slope or rate of change remains a concept difficult to master for students, regardless of the context of the problem (e.g., graphic representation, functional situation) or the perspective held (e.g., geometric or analytical). For example, in those studies and in the present study, students demonstrated difficulty taking into consideration and coordinating both the rise (change in *y*) and the run (change in *x*). As well, students in the present study had difficulty identifying the unit of measurement when asked to determine the slope of a line graphed in a non-homogeneous system of coordinates (MDT problem 7b). This difficulty is confirmed by studies conducted by Zaslavsky et al. (2002). Also of note is that findings of the present study do not support the explanation suggested by Teuscher and Reys (2010) about student difficulties with slope or rate of change, specifically that difficulty with the sign associated with the rise or change in *y* stems from difficulty with the language used in the problem. Only five of the 1561 students in the present study mentioned difficulties with vocabulary, i.e., not understanding the "wording." Notwithstanding that the content of the problems referring to slope or rate of change varied across studies, the findings of the present study and other studies (Orton, 1983; Lobato and Siebert, 2002; Schoenfeld et al. 1993; Stump, 2001a; Zaslavsky et al., 2002) point toward

difficulties with the concept itself, with the mathematical conventions behind graphic representations, and with connections between symbolic and graphic representations.

Another explanation for student difficulties may lie in the way that linear functions, and in particular the concept of slope and the Cartesian connection, are presented and developed in U.S. mathematics curricula. Student difficulties may be associated with the nature of the context that precedes the formal introduction of slope (e.g., ski slopes, roof tops). For example, in a textbook used in one of the school districts participating in this study, slope is introduced to Pre-Algebra students in a physical context followed by a geometric-analytical context:

> In skiing, the term *slope* refers to a slanted mountainside.[…] In math, slope defines the "slant" of a line.[…] For a line on the coordinate plane, slope is the following ratio: $\frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{change in } y}{\text{change in } x}$
>
> (Bennett et al., 2004, p. 545).

In an Algebra I textbook used by the study's participants the physical context is followed by the geometric context of slope:

> Some roofs are steeper than others. In mathematics, a number called *slope* is a measure of the steepness of a line. The *slope of a line* is the ratio of *rise* to *run* for any two points on the line (Rubenstein et al., 1995, p. 361).

The geometric perspective of slope as "rise over run" precedes the functional perspective of slope as rate of change, the latter being introduced in advanced mathematics classes, like Precalculus and Calculus. The results of this study

showed that only students with a geometric perspective of slope had difficulties determining the slope of a graphed line.

One possible explanation for student difficulties with respect to mathematical conventions behind graphing, may also lie in the way graphing is presented in U.S. mathematics curricula. For example, the graph of a function is not defined explicitly in another Algebra I textbooks used by this study's participants. In Larson et al. (2004), students are reminded in Chapter 1 about the procedure to graph a function, and an example is given:

> When you are given the rule for a function, you can prepare to graph the function by making a table showing numbers in the domain and their corresponding output values […] Let the horizontal axis represent the input *t* (in minutes). Label the axis from 0 to 5. Let the vertical axis represent the output *h* (in feet). Label the axis from 0 to 400. Plot the data points given in the table. Finally, connect the points […] (Larson et al., 2004, p. 49).

In this presentation, there is no reference to the units of measurement for the axes (e.g., scale for the *Y*-axis). Difficulty with identifying the scale of the *Y*-axis in non-homogeneous system of coordinates was evident in MDT problem 7b. Students attempted to calculate the "rise" by counting the tick marks "by ones" for the *Y*-axis, instead of recognizing that each interval between two consecutive tick marks was 30 units.

In summary, possible explanations of student difficulties with slope and the Cartesian connection may lie in the way these topics are presented in the school mathematics curriculum.

5.2.5 Research Question 5 Results and Discussion.

Research Question 5: Do teachers' speculations about the nature of students' difficulties correspond to actual student difficulties as identified from student MDT performance and interviews?

To answer Research Question 5, the Mini-Diagnostic Test (MDT) was administered to all participating students, the Ranking Questionnaire for Teachers (RQT) were administered to all participating teachers, and interviews were conducted with 40 students and 20 teachers. Teachers' written comments from the RQT about the nature of their students' difficulties and teachers' interviews were analyzed and compared with students' performance on the MDT and students' interviews.

Teachers, like their students, did not consider the MDT problems as being "hard." About half of the teachers identified the nature of the difficulty as being the "look of the problem," that is the lack of familiarity with the problem (e.g., MDT problem 2), the number of sub-problems in problem (e.g., MDT problem 7 has three sub-problems), or the format of the problem (e.g., MDT problem 1 requires an extended response). These findings are in concert with Lee and Heyworth's (1996, 2000) findings that teachers attend to problem's surface features, rather than to problem's solution requirements to determine problem

difficulty. Of interest is that teachers in the present study solved all the problems before identifying problem difficulty. By contrast, the teachers from Lee and Heyworth's (1996, 2000) study commented on the difficulty without first solving the problems. Despite this difference, teachers' responses were of the same type.

About 42% of the 26 teachers referred to their students' difficulty as being mathematical in nature. Student performance on the MDT and student interviews showed that students had difficulty with MDT problems 1 and 2 referring to the Cartesian connection, and MDT problems 4 and 7b referring to slope. Teachers' speculations about the nature of their students' difficulties did not correspond to actual student difficulties, as identified from student performance on the MDT and student interviews. Only six teachers (23%) mentioned an actual student difficulty in their written comments, missing the Cartesian connection in MDT problem 1, and connecting slope in MDT problem 7b with the rate of change in MDT problem 7c. Teachers' descriptions of student difficulties were mainly at the procedural level (e.g., difficulty substituting $(2,-8)$ for (x_0, y_0) in $y = 3x - 14$). Only four (20%) of the interviewed teachers mentioned actual student difficulties, without being prompted by the researcher. When specifically prompted by the researcher, the percentage of teachers who could describe an actual mathematical difficulty increased overall by about 14%. Mathematical difficulties described by the interviewed teachers were the following: a) difficulty checking whether $(2,-8)$ is a solution to the linear equation $y = 3x - 14$ (MDT problem 1);

b) difficulty finding *x*-values to complete a table of values for the linear function $y = -2x + 3$ (MDT problem 2); c) difficulty estimating the *x*-intercept of a line graphed in a homogeneous system of coordinates (MDT problem 4); and d) difficulty measuring the run or the change in y in a non-homogenous system of coordinates (MDT problem 7b).

Previous research studies reported on teachers' weak pedagogical knowledge of linearity and linear functions (Cunningham, 2005; Davis, 2007; Even & Tirosh, 1995; Stump, 1997, 2001b). When asked to identify student difficulties with slope, some of the teachers from Stump's (1997, 2001b) and Even and Tirosh's (1995) studies mentioned difficulty with the meaning of slope, understanding slope as a measure of steepness, difficulty graphing lines, remembering the slope formula and applying it, and difficulty performing numerical calculations. When prompted by researchers, teachers did not mention student difficulties connecting among various representations (Cunningham, 2005; Davis, 2007; Even & Tirosh, 1995; Stump, 1997, 2001b).

Teachers' weak pedagogical knowledge was mirrored by their weak mathematical knowledge of slope and connections among representations (Stump, 1997, 2001b; McDermott et al., 1987; Orton, 1983; You, 2006; Zaslavsky et al., 2002). For example, teachers had difficulty determining the slope of a line graphed in a non-homogeneous system of coordinates, recognizing slope as a parameter, and as a constant rate of change of two covarying quantities, and connecting between various representations. In the present study, all participating teachers correctly solved the MDT problems. In others studies (Even & Tirosh,

1995; Zhou, 2009) although involving small numbers of teachers, teachers with weak mathematical knowledge, incorrectly solved problems, but could identify some student difficulties and predict student performance. Those studies (Even & Tirosh, 1995; Zhou, 2009) together with the present study, point out that the relationship between mathematical and pedagogical knowledge is complex. In other words, being able to correctly solve a mathematical problem is not associated with being able to predict student difficulties with the problem.

In summary, with respect to students' and teachers' assessments of the nature of problem difficulty, students from Group 1 were better able to describe the mathematical nature of their difficulties than were their teachers. Teachers' assessment of problem difficulty was based on the "look of the problem," that is, student familiarity with the problem, number of sub-problems in the problem, and format of the problem (e.g., extended response, requiring explanations). Without prompting, teachers did not attend to the mathematical nature of actual student difficulties. When prompted by the researcher during interviews, teachers commented on student difficulties regarding the correct application of an algorithm, a formula, or the reading of a graph, and not on conceptual difficulties like conceiving of slope as a measure of steepness in a geometric context. Teachers' weak understanding of student difficulties with important topics like slope or rate of change is even more problematic considering the great amount of instructional time spent teaching the concept. "I beat that topic to death," commented one teacher, surprised that students still had difficulties with slope after being taught for a long period of time (8-10 weeks).

5.3 Limitations of the Study

Limitations of the study focus on the instruments, the study sample, the administration of instruments, and the methods of analysis.

The Instruments. When the MDT was designed, there were two major considerations:

1) The focus should be on a few aspects of linearity and linear functions that, according to previous studies, appeared to be difficult for students in Grades 8-10; and 2) Students must be able to complete the MDT, IQS, and RQS in one class session (i.e., 50 minutes or less).

As a consequence, while providing information on aspects of linearity and linear functions, the MDT is not comprehensive. For example, only two problems focus on slope in geometric–analytical contexts (MDT problems 4 and 7b), two problems address slope as speed (MDT problems 7a and 7c), and three problems focus on the Cartesian connection (MDT problems 1, 2, and 6). It may be that those few problems did not provide a clear picture of the extent of students' knowledge of those concepts. Furthermore, the MDT mean score for Grade 10 students enrolled in Algebra II (N=77) was 8.4, on a scale of 0 to 9, indicating that the problems were too easy for those students, thus creating a ceiling effect (McBee, 2010) and, as a consequence, making it difficult to detect true differences in student knowledge.

The Population. Secondary-level mathematics teachers and their students from two predominately middle-class school districts in the Southwest of U.S. were invited to participate in this study. About two thirds of the participants were

from a school district that had only 2% of students eligible for free or reduced-price lunch. As such, the student population for this study is not representative of the majority of school districts in the United States. For example, in 2007-2008, only 29.6% of the U.S. students were enrolled in secondary schools with less than 25% of students eligible for free or reduced-price lunch (National Center for Education Statistics, 2010).

With respect to current mathematics course, about two-thirds of the participating students were enrolled in courses at their grade level in the mathematics course sequence (Grade 8 - Pre-Algebra, Grade 9 - Algebra I, Grade 10 - Geometry), and one-third of the participating students were enrolled in advanced mathematics courses (i.e., courses above their grade level). Less than 1% of students were enrolled in courses below their grade level. Thus, all results have to be considered in light of average-achieving and high-achieving students.

For the reasons stated above, the findings of this study cannot be generalized beyond the particular population, and the educational context (cultural settings and school curricula).

Administration of Instruments. The MDT and interviews were administered in the middle of the academic year. As a consequence, the timing of administration was not synchronized with the instructional sequences. For example, Pre-Algebra students from one middle school had not yet studied ways to determine the *y*-intercept of a line from the equation of the line (MDT problem 6). If the MDT was administered at a later time, student performance might have been different.

The MDT was administered to students by their teachers. Although they agreed to observe test-like conditions, no supervision of teachers and the test administration took place. Whether or not teachers provided some assistance (e.g., defining a term or clarifying a relationship) is not known.

Finally, interviews were conducted solely by the researcher. As such, probing questions were subject to the researcher's interest. If, instead, the interviews were conducted by a team of researchers, even under the assumption of agreement upon the perspective of looking at the nature of student knowledge and difficulties, and the theoretical framework used for analysis, results might have varied.

Quantitative Methods of Analyses. Data were analyzed using descriptive statistics, including cluster analysis and factor analysis. A regression framework was not used because of the hierarchical or nested structure of the data, i.e., student observations nested within teachers, and thus the violation of the independence assumption necessary for regression. The violation of the assumption of independence may cast doubts upon any statistical significance reported in the study (Raudenbush & Bryk, 2002, pp. 3-10). A hierarchical linear model could not be applied because of insufficient data collected at Level 1 (i.e., student socio-economic status, previous scores on standardized tests).

5.4 Recommendations for Future Research

Based on the limitations described above, recommendations for future research address the assessment instruments, the population, the administration of the MDT and interviews, and the research design.

The Instruments. With respect to the MDT, more problems focusing on the same aspects of linearity and linear functions, but in different contexts (e.g., slope as a measure of steepness in a physical context, slope as rate of change in a covariational context, slope as a parameter in a linear function), would provide a more comprehensive picture of student understanding. Furthermore, including problems of increased complexity, like those recommended by Knuth (2000) and Moschovich et al. (1993), would alleviate the ceiling effects observed in the present study. To avoid ceiling and flooring effects (McBee, 2010), different tests could be administered to different groups of students based on their performance levels, i.e., low-achieving (students enrolled in courses below their grade level), average-achieving (students enrolled in courses corresponding to their grade level), and high-achieving (students enrolled in courses above their grade level).

With respect to the Ranking Questionnaire, participants could be asked to not only rank problems by difficulty, but also, for each problem, identify factors that contributed to the difficulty (e.g., terminology, number of solution steps).

The number of questions in the interviews could be increased to better assess student and teacher familiarity with the problems and problem types. For example, for students, one might ask, "Which of these problems have you seen before? "and "Which problems are new to you?"

For teachers, one might ask, "Which of these problem types have you already explored with your students?"and "Which of these problem types will you explore before the end of the course?"

The Population. Using the instruments and procedures but with a greater cross-section of the general population (e.g., greater variation in socio-economic status) would enhance interpretation and generalization of the findings of the present study.

Administration of Instruments. To ensure that test-like conditions are observed by all participants in the study, supervision by a team of researchers is highly recommended during administration of the MDT and Ranking Questionnaires. To minimize bias in the nature of the probing questions, interviews could be conducted by a team of researchers.

Quantitative Methods of Analyses and Research Design. In the present study, data analyses employed descriptive statistics. Future studies, using Hierarchical Linear Modeling (HLM) will enable investigations of the effects of cross-level data (e.g., curriculum, teaching style) on individual outcomes (e.g., student performance on the MDT). For example, it would be interesting to compare the effects on student performance on the MDT of introducing slope as a rate of change in functional contexts (e.g., constant speed, constant water flow), versus introducing slope as a measure of steepness in a geometric context.

The present study provided a snapshot of student knowledge, and student and teacher perception of student difficulties with linearity and linear functions. Employing a longitudinal study could provide a clear picture of changes in

student knowledge, as well as changes in their teachers' perception of their students' difficulties, over time. This would be valuable information for curriculum developers in the design of mathematics instructional materials. As well, it would be helpful to mathematics educators in the design and conduct of pre-service and in-service programs for middle and high school teachers of mathematics.

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APPENDIX A

INSTRUMENTS

1. The Mini-Diagnostic Test (MDT)

Problem 1

(a) Is the point with coordinates (2,-8) on the graph of the line y = 3x -14?

(b) How did you decide?

Problem 2

y = - 2x + 3

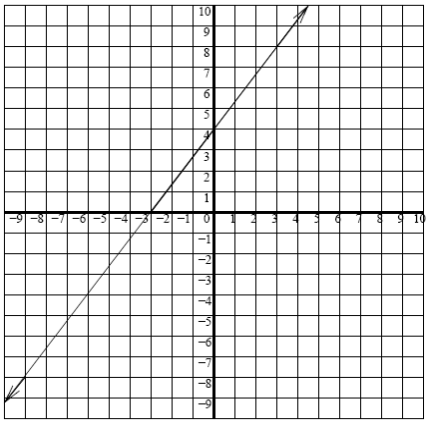
Create a table of values for the equation. Complete 5 rows of the table.

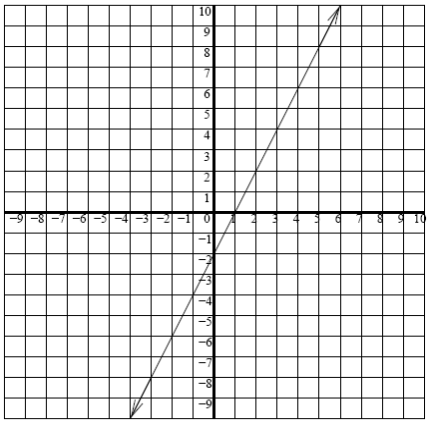
x	y
a.	
b.	
c.	
d.	
e.	

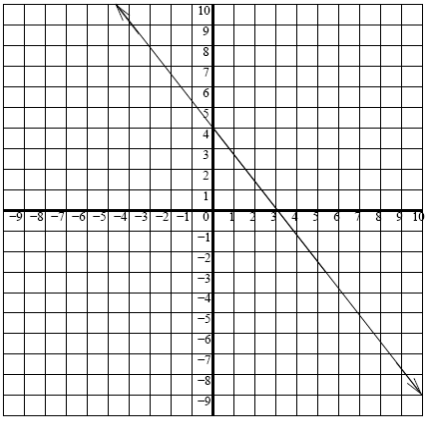
Problem 3

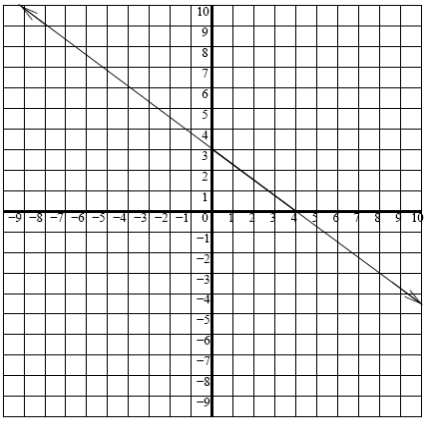
Which of the graphs below contains the points given in the table?

x	y
3	4
6	10
9	16
12	22

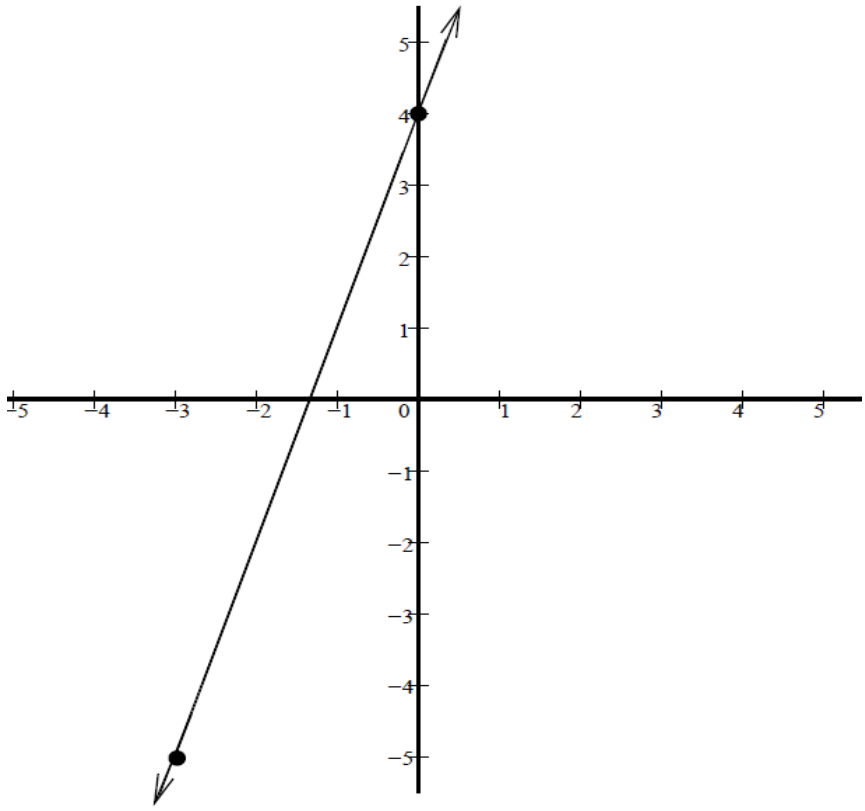
a) 

b) 

c) 

d) 

Problem 4



In the graph above, what is the slope of the line? Show your work.

Problem 5

The owner of a car dealership noticed a pattern in the weekly car sales, as shown in the table below.

Weekly Car Sales

Week (w)	Number of Cars Sold (s)
1	12
2	18
3	24
4	30

Which of the following equations represents the relationship between the number of cars sold (s) and the number of weeks (w)?

(a) s = 6w; (b) s = 12w; (c) s = 6w + 6; (d) s = 6w + 12

Problem 6

Given this linear equation 2x + 3y = 12, which of the following is the y-intercept?

(a) 0; (b) 2; (c) 4; (d) 6; (e) 12

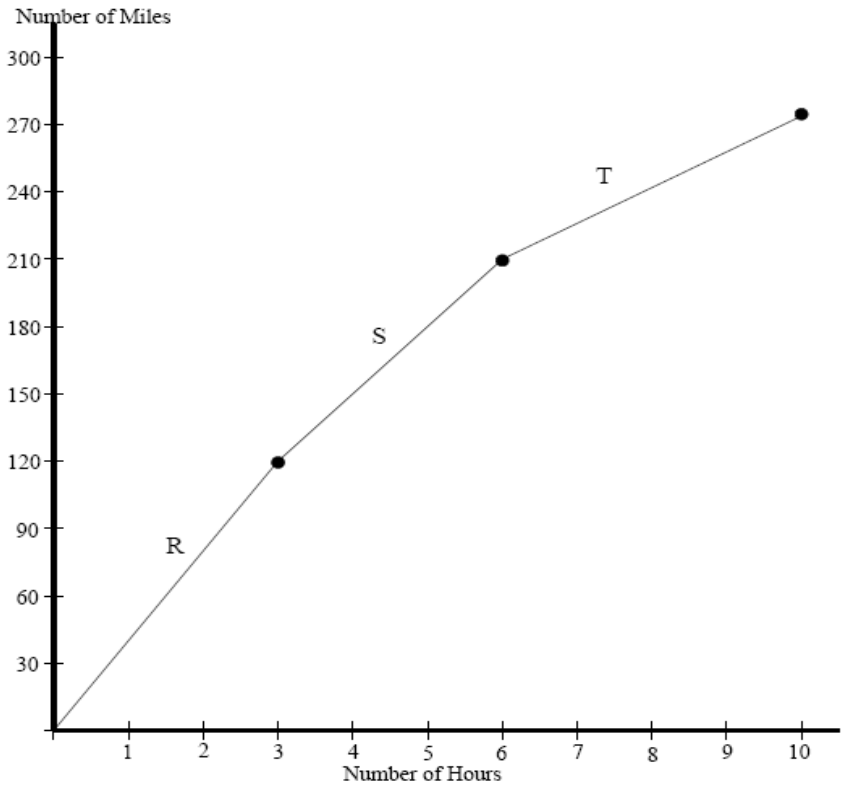
Problem 7

The graph below represents the distance that a car traveled after different numbers of hours.

(a) Which part of the graph(R, S, or T) represents the hours when the car moved the slowest?

(b) What is the slope of part R of the graph?

(c) What is the speed of the car in part R?



2. Information Questionnaire for Students

Tell us about yourself:

1. Name (print)

_____

2. Grade (8, 9 or 10) _____

3. Name of your math course (e.g., Pre-Algebra, Algebra I, Geometry)

_____

4. Name of your math teacher

_____

3. Ranking Questionnaire for Students

1. Solve the seven problems.
2. Show all of your work on the pages.
3. When you finish the problems, please complete the table.

    Tell us which problem was most difficult.

    Then write the other numbers of the problems in order of difficulty.

Difficulty Level	**Problem Number**
Most difficult	
Least difficult	

4. For the problem you named the most difficult, tell us why you chose it.

4. Information Questionnaire for Teachers

Information About Yourself

1. Highest college degree?
2. College major?
3. Titles of courses you are currently teaching.
4. Titles of the courses you have taught in the past 2 years.
5. Number of years you have been teaching?
6. Name (print)
7. School:
8. Phone:
9. Email:

5. **Ranking Questionnaire for Teachers**

1. Solve the seven problems.
2. Which of the problems will be most difficult for students? Least difficult?

   Rank order the problems by difficulty.

   Use the numbers of the problems.

   Complete the table.

Difficulty Level	**Problem Number**
Most difficult for students	
Least difficult for students	

3. For the two problems that you identified as most difficult for your students, describe the student difficulty.

   a) Most Difficult:

   b) Next Most Difficult:

APPENDIX B

SCORING GUIDE AND CODING MANUAL

PART I: Scoring Guide (SG)

To score the MDT, use Table 51. Score each of the parts of MDT problem 1 (part 1a and part 1b) with 1 for a correct answer and 0 otherwise. The total score for MDT problem 1 is obtained by adding the scores for parts 1a and 1b and dividing the sum by 2. Score MDT problem 2 with 0, .2, .4, .6, .8 or 1, based on the number of correctly completed rows (0, 1, 2, 3, 4, or 5) in the table of values. Score MDT problems 3, 4, 5, 6, 7a, and 7b with 1 for a correct response and 0 otherwise. MDT problem 7c is separated into two parts, 7c1 and 7c2, one part for the numeric value and the other for the unit, with 1 point for each correct answer and 0 otherwise. The total score for MDT problem 7c is obtained by adding the two scores for parts 7c1 and 7c2 and dividing the sum by 2.

Table 51

Scoring Guide for MDT Scores by Problem

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Problem 7		
						a	b	c
0, .5 or 1	0, .2, .4, .6, .8 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0, .5, or 1

Use Tables 52-60 to record the frequencies (number of students) for the types of responses for each MDT problem.

Table 52

Scoring Guide for MDT Problem 1 Types of Responses

PART	Types of Answer	Number of Students
PART (a)	1. *Correct Answer "Yes"* 2. Answer "No" 3. No answer or an answer different from "Yes" or "No"	
PART (b)	1. *Correct answer using the Cartesian Connection* Identifying $(2,-8)$ as (x_0, y_0) and substituting in $y = 3x - 14$ *;* and Checking if the statement $-8 = 3(2) - 14$ (or equivalent statement) is true 2. Incorrect or incomplete answer while attempting to use the Cartesian connection 3. *Correct answer using a graph* Graphing the line $y = 3x - 14$ (using a quick graph or a table of values); Plotting the point (2, -8); and Inspecting to see if the point is on the line 4. Incomplete or incorrect answer from use of a graph 5. Incomplete or incorrect responses different from those above from Part (b), #1-4 6. No answer or wrote "I guessed"	

Table 53

Scoring Guide for MDT Problem 2 Types of Responses

Types of Answers	Number of Students
1. *5 rows correctly completed*	
2. 4 rows correctly completed	
3. 3 rows correctly completed	
4. 2 rows correctly completed	
5. 1 row correctly completed	
6. No row correctly completed, but attempted to answer	
7. No answer	

Table 54

Scoring Guide for MDT Problem 3 Types of Responses

Types of Answers	Number of Students
1. a	
2. *Correct answer b*	
3. c	
4. d	

Table 55

Scoring Guide for MDT Problem 4 Types of Responses

Types of Answers	Number of Students
1. *Correct answer "3"*	
2. Computational errors in calculation of slope, but correct formula	
3. Reversed ratio	
4. Incorrect answer different from answers to 2 and 3	
5. Irrelevant information	
6. No answer or answer, "I don't know."	

Table 56

Scoring Guide for MDT Problem 5 Types of Responses

Types of Answers	Number of Students
1. a	
2. b	
3. *Correct answer c*	
4. d	

.

Table 57

Scoring Guide for MDT Problem 6 Types of Responses

Types of Answers	Number of Students
1. a	
2. b	
3. *Correct answer c*	
4. d	

Table 58

Scoring Guide for MDT Problem 7a Types of Responses

Types of Answers	Number of Students
1. Answer "R"	
2. Answer "S"	
3. ***Correct** answer "T"*	
4. Answer different "R", "S", or "T "	
5. No answer	

Table 59

Scoring Guide for MDT Problem 7b Type of Responses

Types of Answers	Number of Students
1. Correct answer "40"	
2. Incorrect answer	
3. No answer or irrelevant information	

Table 60

Scoring Guide for MDT Problem 7c Types of Responses

Types of Answers	Number of Students
1. Correct answer "40mph"	
2. Correct numeric answer, but incorrect or no unit of measurement given	
3. Incorrect answer	
4. No answer or irrelevant information	

PART II: Coding the Written Comments from Ranking Questionnaires

After solving the MDT problems, all students identify the most difficult problem and explain why it is difficult. Teachers identify the two most difficult problems for their students, and describe their suspicions about why students found the problem difficult.

Code student and teacher comments on the Ranking Questionnaires using the categories *H, L, M, R, T, U* shown in Tables 61-66. Note: Each table contains a description of the category and examples of what constitutes evidence for its presence, for both students and teachers.

Table 61

Category H: Hard Problem

Description: A general statement about the difficulty of a problem, topic or domain. Sometimes a mathematical concept is identified as "hard" or "difficult." Other times a reference to unsatisfactory performance is made, but no specific details about the nature of the difficulty are given.	
Examples from Students' Written Comments	*Examples from Teachers' Written Comments*
"It was a difficult question."	"This problem is hard for my students."
"I thought that one was the hardest was because those types are just hard to me."	"Slope is difficult for student."
"I just really have a hard time on slopes."	

Table 62

Category L: Never Learned /Taught

Description: A statement about the novelty of a problem, or that a mathematical concept or how to solve a certain problem was not learned or taught.	
Examples from Students' Written Comments	*Examples from Teachers' Written Comments*
"Problem number 1 was the most difficult because we have not learned this yet."	"Haven't seen the material."
"I have never seen a problem like that before."	"Graph interpretation , application of skills for graphing (not learned art this time)."
"I was absent when the teacher taught it."	"We hadn't explained how to find slope."

Table 63

Category M: Mathematical Difficulty

Description: Nature of difficulty refers to lack of understanding of mathematical concepts, or their representations, or difficulties with algorithms/procedures, computations, and symbolic manipulations.	
Examples from Students' Written Comments	*Examples from Teachers' Written Comments*
"I didn't understand at first what *x* and *y* were supposed to be. I drew out a graph and then it just clicked in my head the *x* and *y* axis." "2 was the most difficult because you had to figure out what numbers you had to put in because it doesn't tell you." "4, it wasn't a grid to help." "Because it was hard to determine the exact numbers the dots were placed."	"If the students plug the #s into the equation, they might know what to do next –simplify, then they may not know how to check if it is 'true' or not." "Students struggle with getting to make up values for *x*…they are used to being given the numbers and then solving." "No points labeled." "Using numbers so high."

Table 64

Category R: Not Remembering

Description: A statement about forgetting or not remembering the mathematical concepts or how to solve a problem.	
Examples from Students' Written Comments	*Examples from Teachers' Written Comments*
"I forgot how to do this type." "I didn't remember how to do it." "I couldn't remember the formula for slope."	"Students forget how to find slope." "Students forget what slope means." "They don't know where to start."

Table 65

Category T: Thinking Time

Description: A reference to the amount of time, thinking, memory or any other resource necessary to solve a problem, or to a problem feature, as for example the number of subparts to be completed.	
Examples from Students' Written Comments	*Examples from teachers' Written Comments*
"Had to do the most work." "You had to find 5 values which took time." "It was the problem I had to think on." "It required the biggest formula." "It took a long time for one problem."	"However, they do have difficulty explaining how they decide things." "Lots of reading."

Table 66

Category U: Not Understanding

Description: A general statement about not understanding a mathematical concept, topic or domain, how to solve a problem, or the vocabulary used, without offering any further details about the nature of the difficulty.	
Examples from Students' Written Comments	*Examples from Teachers' Written Comments*
"I didn't know it." "Your wording was weird and I just didn't understand it!" "No understandings of slope."	"My students can solve it but the wording is not clear." "Students never get it."

Part II: Coding the Interviews

Students and teachers solve the MDT problems aloud, and comment on the nature of difficulties for only MDT problems 1, 2, 4 and 7b.

Conceptual Knowledge of the Cartesian Connection (CKC) is inferred from strategies used by students to solve MDT problems 1 and 2, and from teachers' answers to two questions: "What do your students need to know in order to solve this problem?" and "How would you help your students overcome difficulties related to this problem?"

Table 67 presents CKC subcategories and samples of student and teacher comments to assist in the coding.

Table 67

Conceptual Knowledge of the Cartesian Connection (CKC) and its Subcategories

CKC Subcategories	Examples from: A. Students' Interviews B. Teachers' Interviews
CKC-D From Point to Line A point (x_0, y_0) is on the graph of the line *l* if its coordinates satisfy the equation of *l*, $y = mx + b$, i.e. if the statement $y_0 = mx_0 + b$ is a true statement.	A. "I will substitute in the *x* value for 2 and the *y* value as -8. So my equation would now be $-8 = 3(2) - 14$. First I multiply 3 by 2 and that equals 6 and I subtract, 6 minus 14 and that is -8, and -8 equals -8. That's a check so I would say 'yes'." B. "They need to recognize that points should make this equation true and if it is true then the point is on the line and if it's not true the point's not on the line."
CKC-C From Line to Point(s) If a point is (x_0, y_0) on the graph of the line *l* then its coordinates satisfy the equation of *l*, $y = mx + b$. There is an infinite number of points (x_0, y_0) on the line *l* satisfying the equations of the line *l*, $y = mx + b$.	A."I just picked random numbers for *x* and then I plugged the numbers I chose into the equation and got the *y* values." B."You have a situation where you have an infinite amount of solutions and those solutions are all ordered pairs."

The Conceptual Knowledge of Slope (CKS) is inferred from students' answers to the question, "What do you mean by slope?" and from their explanations given while solving problems 4, and 7b. Teachers' knowledge of slope is inferred from interview questions: "How would you explain what slope means to your students?" and "What would you do to help your students overcome these difficulties (e.g., with slope)?"

CKS subcategories and samples of student and teacher comments to assist in the coding are presented in Table 68.

Table 68

Conceptual Knowledge of Slope (CKS) and its Subcategories

CKS Subcategories	Examples from: A. Students' Interviews B. Teachers' Interviews
CKS-P Physical Context Slope describes the steepness of physical objects such as mountain roads, sky slopes, and wheelchair ramps. A statement about an entire class of objects sharing the property of having a slope or being steep, or an instantiation/particular example given.	A."Slope is like the elevation of the line." B. "Like when you have a ladder, if you are leaning it up against the wall and when you go down the slide. A roof has a slant that slant is the slope of the roof."
CKS-G Geometric-Analytical Context Given two points (x_1, y_1) and (x_2, y_2), the slope of the line connecting the two points is "rise over run," or $m = \frac{y_2 - y_1}{x_2 - x_1}$. Homogeneous Systems of Coordinates: Slope is a geometric property of the line $y = mx + b$ as well as an analytical property of the linear function $y = mx + b$. Non-homogeneous Systems of Coordinates: Slope is an analytical property of the function $y = mx + b$.	A. "To find the slope of a line I would do the rise over the run. I actually don't remember how to find the slope with just an equation or formula right now, so I'm just going to go ahead and count… the rise is 1, 2, 3, 4, 5, 6, 7, 8, 9 and the run is 1, 2, 3, I believe. So it would be 9 over 3 because rise over run, or reduced, would be a slope of positive 3." B. "The the rise over the run using the formula that we've given to them, $\frac{y_2 - y_1}{x_2 - x_1}$, which is the slope formula."

Student Difficulties with the Cartesian Connection from Point to Line (SDC-D) are inferred from student schemes and teacher comments, and occur at each of the steps of the algorithm used to solve MDT problem 1.

Table 69 describes the SDC-D subcategories and presents samples of student and teacher comments in each subcategory.

Table 69

Student Difficulties with Cartesian Connection from Point to Line SDC-D (MDT Problem 1)

SDC-D Subcategories	Examples from: A. Students' Interviews B. Teachers' Interviews
SDC-DP Identifying $(2,-8)$ as (x_0, y_0) and replacing variables with their values in $y = 3x - 14$.	A. " I'm not too sure about this but, I'd say that… like that." (*Researcher Note*: *The student wrote* $y = 3(2,-8) - 14$)
	B. "They have a problem of substituting, basically, and knowing an ordered pair. Knowing that *x* comes first, *y* comes second and knowing that they are substituting in these values into this equation."
SDC-DT Checking if the statement $-8 = 3(2) - 14$ (or other equivalent statement) is true .	A. "You take this equation $y = 3x - 14$ and you plug in what you got right here so -8 equals 3times 2 minus 14 is going to be right here so you get $-8 = 6 - 14$ and then from there wait, hold on, oh, I forgot how to do this."
	B. "They need to know that if the coordinates $(2,-8)$ are in the graph of that line, then they should fulfill that equation, so if you put them in, it should balance it, it should be equal on both sides."

Student Difficulties with the Cartesian Connection from Line to Points (SDC-C) are inferred from student scheme and teacher comments, and occur at each of the steps of the algorithm used to solve MDT problem 2.

Table 70 describes SDC-C subcategories and presents examples of student and teacher comments to assist in the coding.

Table 70

Student Difficulties with Cartesian Connection from Line to Point SDC-C (MDT Problem 2)

SDC-C Subcategories	Examples from: A. Students' Interviews B. Teachers' Interviews
SDC-CX Identifying *x*-values	A. " I think you go and plug or for *x* you put 0, -1, or you put a negative, 0, a positive and then fractions so you can get all of them, see, if you were going to put and use I think but I don't know…" B."…It's sort of open-ended in that they can fill in any value for *x* …just because of how open-ended it was. If the problem included values for *x*, it might be a little bit less difficult for them."
SDC-CY Finding corresponding *y*-values	A."I would then go on and do is find out my *y* coordinates by replacing *x* with 0, and then continuing on from there and so after I replace *x* with 0 I get that $y = -3$ …"(Researcher Note: She gives values to *x* and solves for *y*; then she gives values to *y* and solves for *x*. She records these values in her table, but they do not satisfy simultaneously the equation $y = -2x + 3$). " B. "Finding *y* is going to be difficult, and filling out that table."

Student Difficulties with Slope (SDS) are inferred from student answers and teachers' comments on difficulties with slope. Only difficulties with slope in geometric context, in MDT problem 4 and MDT problem 7b, are coded.

Table 71 presents the subcategories for Student Difficulties with Slope in an Homogenous System of Coordinates (SDS-H), accompanied by samples of student and teacher comments to assist in the coding.

Table 71

Student Difficulties with Slope in Homogeneous Systems of Coordinates SDS-H (MDT Problem 4)

SDS–H Subcategories	Examples from: A. Students' Interviews B. Teachers' Interviews
SDS- HR Calculating slope as a ratio	A. "I can't remember if I'm supposed to add 9 plus 3 or multiply or divide." B. "Students don't know what to do with the rise and the run. They don't know they have to divide because slope is a ratio."
SDS-HM Identifying the origin from where the measurement should start (e.g., considering the origin of the system of coordinates as starting point in measuring the rise)	A. "Slope is the rise over run it goes up four and it doesn't go in any direction left or right, it's 4." B."Students don't know where to start counting for the rise. They think they should start from the origin."
Identifying the unit of measurement (e.g., estimating non-integer coordinates like the *x*-intercept≈ 1.5)	A. "And the run, I just counted how many it went over until the line intersected it. So it would be one and a half. " B." If we refer to the point where this line from problem 4 intersects *X*-axis, they could give you many different values. They could give you 0, they would tell you that it's (0, 1.5)…"

Table 72 presents the subcategories for Student Difficulties with Slope in a Non-Homogenous System of Coordinates (SDS-N) and samples from student and teacher comments to assist in the coding.

Table 72

Student Difficulties with Slope in Non-Homogeneous Systems of Coordinates

SDS-N (MDT Problem 7b)

SDS-N Subcategories	Examples from: A. Students' Interviews B. Teachers' Interviews
SDS-NM Considering slope as a geometric property of the line, and not an analytical property of the function in non-homogeneous systems of coordinates from problem 7.	A. "I don't know if you have to take these numbers into like, consideration, if you count it as 30, as go up 30 or 4 over 3 but it doesn't really make sense because that means you'd have to go up 4, oh no, it does make sense … ah, I don't know. " B. "They're counting the rise over the run. So they're going over, they know how many spaces they're going over, but they're counting the spaces, they're not counting the quantity. So they're going over 3 like they should, they know that they're going over 4, but they're not associating 4 with this value."

APPENDIX C
RECRUITMENT LETTERS AND IRB APPROVALS

1. Letter to Students

Dear Student:

We are conducting a study about students' understanding of some topics in algebra in order to discover better ways to support learning.

I am inviting you to participate in the study by completing a short 7-item quiz about beginning ideas from algebra. The quiz will take about 30-40 minutes to complete.

Your participation in this study is voluntary. The results will not affect your grade. If you choose not to participate or to withdraw from the study at any time, your course grade will not be affected. Your work on the quiz will remain anonymous.

If you agree to participate in this study, please complete the information below. You also will need to have your parent or guardian sign an agreement form as well.

Return the signed part of the form to your teacher.

If you have any questions about this study, please contact me at the email or phone number listed below.

Thanks in advance for your help.

Valentina Postelnicu
ASUP Phys Chem Applied Math
(480)727-1977
e-mail: Valentina.Postelnicu@asu.edu

If you have any questions about your participation in this research, or if you feel you have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Research Compliance Office at (480) 965-6788.

I, _____, agree to participate in the algebra study.
(Printed Name)

_____ _____
(Signature) (Date)

2. Parental Consent for Interviewing Minors

Dear Parent or Guardian,

The School of Educational Innovation and Teacher Preparation at Arizona State University is focusing on ways to help children learn mathematics. One of the topics that is often difficult for students is algebra. As part of this study, your child completed a short quiz. You signed a consent form for that activity. To gain greater insight into students' reasoning processes, we would like your permission to have your child talk about how some of the problems were solved. This "think aloud" will be audio taped, and will take about 20 minutes to complete. The taping will take place during school time. Your child's name will not be used during the taping and will not be used in the transcription. Only the research team will have access to the audiotape. At the end of our study, the tape will be destroyed.

Your child's participation in this study is voluntary. If your child chooses not to participate or to withdraw from the study at any time it will not affect the course grade.

If you would like additional information, please contact me or Valentina Postelnicu using the contact information below.

Thank you again for assisting with this important study.

With best wishes,

Carole Greenes, Ed.D.
Dean and Professor Mathematics Education, School of Educational Innovation and Teacher Preparation, Arizona State University (480) 727-1820,
e-mail: Carole.Greenes@asu.edu

Valentina Postelnicu
Instructor, ASUP Phys Chem Applied Math
(480)727-1977, e-mail: Valentina.Postelnicu@asu.edu

If you consent to your child's participation in this study, please complete the section below and have your child return this form to the teacher as soon as possible. A copy of this signed form will be returned to you. Thank you.

_____ to be audio taped for this study.

Student Name

_____ _____ _____

Parent Signature Parent Name Date

If you have any questions about you or your child's rights as a subject/participant in this research, or if you feel you or your child have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Research Compliance Office at (480) 965-6788.

3. Letter to Students for Interview

Dear Student:

We are conducting a study about students' understanding of some topics in algebra in order to discover better ways to support learning.

I am inviting you to be one of 10 students from your school to talk about how you solved some of the problems on the algebra quiz that you already completed. The interview will last about 20 minutes, will take place during school time, and will be audio taped.

Your participation in this part of the study is voluntary and the results will not affect your course grade. You may choose not to participate in or to withdraw from the taping at any time. Again, that decision will not affect your course grade. Transcriptions of the taping will not use your name. The audiotape will be destroyed at the end of this study.

If you agree to be tape recorded, please complete the information below. You also will need to have your parent or guardian sign an agreement form as well. *Return the signed part of the form to your teacher.*

If you have any questions about this study, please contact me at the email or phone number listed below.

Thanks in advance for your help.

Valentina Postelnicu
ASUP Phys Chem Applied Math
(480)727-1977, e-mail: Valentina.Postelnicu@asu.edu

If you have any questions about your participation in this research, or if you feel you have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Research Compliance Office at (480) 965-6788.

I, _____, agree to be taped while I solve algebra problems.

(Printed Name)

_____ _____

(Signature) (Date)

4. Letter to Teachers

Dear Colleague,

We need your help. The School of Educational Innovation and Teacher Preparation at Arizona State University, Polytechnic campus, is focusing on ways to enhance student achievement in mathematics. As a first step, we examined student performance on state and other standardized tests. This analysis revealed that students in Grades 8-10 have particular difficulty with some fundamental topics in algebra. To determine how well our instructional programs address these important concepts, we developed Mini-Diagnostic Tools, (MDT). The first of these, MDT -1, assesses students' understanding of lines, how lines are represented with words, tables, graphs and symbols. MDT-1 consists of 7 problems with formats much like those used in the AIMS test.

MDT-1 can be administered to students in one class period. MDT-1 should take no more than 35 minutes for its completion. We would like you to administer MDT-1 to your Grade 8, 9 or 10 students.

To gain clearer understanding of students' solution methods, after the administration of MDT-1, we would like to conduct interviews with you and 10 students from your school . Interviews will be approximately 20-30 minutes in length and can be conducted at the school sites. Interviews will require students who completed the quiz to orally recount their solution processes to some of the items. Interviewers will be members of the project staff, all of whom are experienced teachers and are either faculty or graduate students in mathematics education in the School of Educational Innovation and Teacher Preparation.

Prior to the administration of the MDT-1, signed permission forms must be received from all students and their parents or guardians. For the audio-taped oral recounting of solution processes, we will need a separate form signed by parents or guardians. No teacher or student names will be collected. Audio tapes will be shared only with the project staff and will be destroyed after the completion of the project.

If you would like additional information, please contact me or Valentina Postelnicu. Email is the best way to reach us.

With great appreciation and best wishes,

Carole Greenes, Ed.D.
Dean and Professor Mathematics Education, School of Educational Innovation and Teacher Preparation, Arizona State University at the Polytechnic campus (480) 727-1820,
e-mail: Carole.Greenes@asu.edu
Valentina Postelnicu
Instructor, ASUP Phys Chem Applied Math, (480)727-1977,
e-mail: Valentina.Postelnicu@asu.edu

5. IRB Approvals

ASU ARIZONA STATE UNIVERSITY
Research Compliance Office
Office for Research & Sponsored Projects Administration
P.O. Box 873503
Tempe, AZ 85287-3503
Phone
(480) 965-6788
Facsimile
(480) 965-7772

ASU ARIZONA STATE UNIVERSITY
RESEARCH AND ECONOMIC AFFAIRS

Office of Research Integrity and Assurance

To:	Carole Greenes
From:	Mark Roosa, Chair Soc Beh IRB
Date:	11/13/2008
Committee Action:	**Exemption Granted**
IRB Action Date:	11/13/2008
IRB Protocol #:	0810003402
Study Title:	Prime the Pipeline Project (P3): Putting Knowledge to Work

The above-referenced protocol is considered exempt after review by the Institutional Review Board pursuant to Federal regulations, 45 CFR Part 46.101(b)(1) .

This part of the federal regulations requires that the information be recorded by investigators in such a manner that subjects cannot be identified, directly or through identifiers linked to the subjects. It is necessary that the information obtained not be such that if disclosed outside the research, it could reasonably place the subjects at risk of criminal or civil liability, or be damaging to the subjects' financial standing, employability, or reputation.

You should retain a copy of this letter for your records.

ARIZONA STATE UNIVERSITY
Research Compliance Office
Office for Research & Sponsored Projects Administration
P.O. Box 873503
Tempe, AZ 85287-3503
Phone
(480) 965-6788
Facsimile
(480) 965-7772

To:	Carole Greenes
From:	Mark Roosa, Chair Soc Beh IRB
Date:	06/12/2008
Committee Action:	**Exemption Granted**
IRB Action Date:	06/12/2008
IRB Protocol #:	0806003040
Study Title:	The High School Students" Understanding of Linear Functions

The above-referenced protocol is considered exempt after review by the Institutional Review Board pursuant to Federal regulations, 45 CFR Part 46.101(b)(1) .

This part of the federal regulations requires that the information be recorded by investigators in such a manner that subjects cannot be identified, directly or through identifiers linked to the subjects. It is necessary that the information obtained not be such that if disclosed outside the research, it could reasonably place the subjects at risk of criminal or civil liability, or be damaging to the subjects' financial standing, employability, or reputation.

You should retain a copy of this letter for your records.

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Human Participant Protections Education for Research T

Completion Certificate

This is to certify that

Valentina Postelnicu

has completed the **Human Participants Protection Education for Research Teams** online course, sponsored by the National Institutes of Health (NIH), on 01/28/2007.

This course included the following:

- key historical events and current issues that impact guidelines and legislation on human participant protection in research.
- ethical principles and guidelines that should assist in resolving the ethical issues inherent in the conduct of research with human participants.
- the use of key ethical principles and federal regulations to protect human participants at various stages in the research process.
- a description of guidelines for the protection of special populations in research.
- a definition of informed consent and components necessary for a valid consent.
- a description of the role of the IRB in the research process.
- the roles, responsibilities, and interactions of federal agencies, institutions, and researchers in conducting research with human participants.

National Institutes of Health
http://www.nih.gov