

Relativistic Fermions in a Magnetic Field:
From Quantum Hall Effect in Graphene to Chiral Asymmetry in QED

by
Lifang Xia

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved April 2016 by the
Graduate Supervisory Committee:

Igor Shovkovy, Chair
Richard Lebed
Kevin Schmidt
Easson Damien

ARIZONA STATE UNIVERSITY

May 2016

ABSTRACT

In the first part of this thesis, we use the generalized Landau-level representation to study the effect of screening on the properties of the graphene quantum Hall states with integer filling factors. The analysis is performed in the low-energy Dirac model in the mean-field approximation, in which the long-range Coulomb interaction is modified by the one-loop static screening effects. The solutions demonstrate that static screening leads to a substantial suppression of the gap parameters in the quantum Hall states with a broken $U(4)$ flavor symmetry. The results of the temperature dependence of the energy gaps mimic well the temperature dependence of the activation energies measured in experiment. The Landau-level running of the quasiparticle dynamical parameters could be tested via optical studies of the integer quantum Hall states.

In the second part, by using the generalized Landau-level representation, we study the interaction induced chiral asymmetry in cold QED plasma beyond the weak-field approximation. The chiral shift and the parity-even chiral chemical potential function are obtained numerically and are found peaking near the Fermi surface and increases and decreases with the Landau level index, respectively. The results are used to quantify the chiral asymmetry of the Fermi surface in dense QED matter. The chiral asymmetry appears to be rather small even in the strongest magnetic fields and at the highest stellar densities. However, the analogous asymmetry can be substantial in the case of dense quark matter.

I dedicate this thesis to all who have helped and supported me and are in my life.

ACKNOWLEDGMENTS

First, I thank greatly my advisor, Dr. Igor Shovkovy, for his guidance, help, and patience in my study and research. He taught me knowledge, shared his ideas in physics, and directed me into the field. He has been a great support to me both in research and in ordinary life and encouraged me greatly. A big thanks is due to him.

I also thank deeply my advisors in astrophysics, Dr. Sangeeta Malhotra, Dr. James Rhoads and Dr. Xu Zhou for their trust, care and support at my difficult times.

I would also like to thank my committee members: Prof. Richard Lebed, Prof. Kevin Schmidt, and Prof. Damien Easson, for their valuable time and supervisions to this work.

I am also thankful to all faculty, staff and classmates in the Physics Department at Arizona State University and all friends for their assistance, support and friendship.

Finally, I would like to thank my family, my father, my mother, and my brother, for their always understanding, love, encouragement and support, and my little nephew, bringing me so much joy.

My research was supported in part by the U.S. National Science Foundation under Grant No. PHY-1404232.

TABLE OF CONTENTS

	Page
LIST OF TABLES	vi
LIST OF FIGURES	vii
CHAPTER	
1 INTRODUCTION	1
1.1 Magnetic Catalysis	2
1.1.1 Magnetic Catalysis in 2+1 Dimensions	2
1.1.2 Magnetic Catalysis in 3+1 Dimensions	5
1.1.3 Magnetic Catalysis in QED	6
1.2 Quantum Hall Effect in Graphene	7
1.2.1 Dirac Fermions in Strong Magnetic Field	7
1.2.2 Graphene QHE: Experiments	11
1.2.3 Graphene QHE: Theory	14
1.3 Chiral Asymmetry in a Magnetic Field	16
1.3.1 Chiral Separation and Chiral Magnetic Effects	17
1.3.2 Chiral Asymmetry in a Weak Magnetic Field	17
2 GRAPHENE QUANTUM HALL EFFECT IN STATIC SCREENING	23
2.1 Motivation	23
2.2 Model	27
2.2.1 Quasiparticle Hamiltonian	28
2.2.2 Full Quasiparticle Propagator	30
2.2.3 Schwinger-Dyson Equation with Static Screening	33
2.3 Numerical Results	38
2.3.1 Fermi Velocity Renormalization in Weak Magnetic Field	40

CHAPTER	Page
2.3.2 Quantum Hall States in Strong Magnetic Field	43
2.3.3 Temperature Dependence of the Energy Gaps	53
2.4 Discussion	56
3 CHIRAL ASYMMETRY IN QED IN A STRONG MAGNETIC FIELD	63
3.1 Motivation	63
3.2 Model	67
3.2.1 Fermion Propagator	67
3.2.2 Fermion Self-energy	68
3.2.3 Chiral Shift and Chiral Chemical Potential	70
3.3 Numerical Results	72
3.3.1 Monte Carlo Importance Sampling	73
3.3.2 Chiral Shift	76
3.3.3 Chiral Chemical Potential	78
3.3.4 Chiral Asymmetry	80
3.4 Discussion	81
4 SUMMARY AND OUTLOOK	83
4.1 Summary	83
4.2 Outlook	85
REFERENCES	87
APPENDIX	
A FERMION PROPAGATOR WITH CHIRAL ASYMMETRY	95
B CHIRAL ASYMMETRY FUNCTIONS IN THE HDL APPROXIMATION	99

LIST OF TABLES

Table	Page
2.1 Kernel Coefficients $\kappa_{m,n}^{(0)}$ with Effects of Static Screening Included.	37
2.2 Kernel Coefficients $\kappa_{m,n}^{(1)}$ with Effects of Static Screening Included.	38
2.3 Wavefunction Renormalization f_n for Several Values of Chemical Potentials with Screening Effects Considered, Compared with Solutions with Screening Effects Neglected.	41
2.4 Parameters $C_{n,n'}$ in QH State with Filling Factor $\nu = 2$ in Model with Static Screening.	43
2.5 Parameters $C_{n,n'}$ in QH State with Filling Factor $\nu = 6$ in Model with Static Screening.	43
2.6 Gap Parameters for Solutions When Fermi Energy Close to the Lowest Landau level.	46
2.7 Gap Parameters for Solutions When Fermi Energy Close to the Landau level at $n = 1$	49
3.1 Chiral Asymmetry Functions Δ_n , $\mu_{5,n}$, and $(p_3 - p_3^{(0)})/\mu$ at Fermi Surface. . . .	82

LIST OF FIGURES

Figure	Page
1.1 Lattice Structure of Graphene in Coordinate Space and the Brillouin Zone in Reciprocal Space.	8
1.2 Band Structure of Graphene Electrons at Dirac Points.	9
1.3 Quantum Hall Effect in Graphene: Observed Plateaux at Filling Factors $\nu = 4(n + 1/2)$ in Hall Conductivity σ_{xy} of Graphene.	12
1.4 Quantum Hall effect in Graphene: Newly Observed Plateaus at Filling Factors $\nu = 0, \pm 1, \pm 4$ at Magnetic Fields $B > 25$ T.	13
2.1 Dependence of Static Polarization Function $\Pi(0, k)$ on Dimensionless Vairable kl	36
2.2 Kernel Coefficients $\kappa_{m,n}^{(0)}$ and $\kappa_{m,n}^{(1)}$ with Effects of Static Screening Included. . .	37
2.3 Free Energies of Several Lowest Energy Solutions at the Lowest Landau level. .	44
2.4 Wavefunction Renormalization Parameters $f_{n,s}$ as Functions of Landau Level index for Four Different Type Filling Factors of QH States.	50
2.5 Chemical Potential Differences $(\mu_{n,s} - \mu_s)$ as Functions of Landau Level index for Four Different Type Filling Factors of QH States.	51
2.6 Singlet Haldane Masses $\Delta_{n,s}$ as Functions of Landau Level index for Four Different Type Filling Factors of QH States.	52
2.7 Triplet Chemical Potentials $\tilde{\mu}_{n,s}$ and Dirac Masses $\tilde{\Delta}_{n,s}$ as Functions of Landau Level index for Different Type Filling Factors of QH States.	53
2.8 Energy Gaps as Functions of Temperature for QH States with Filling Factors $\nu = 4n \pm 1$	55
2.9 Energy Gaps as Functions of Temperature for QH States with Filling Factors $\nu = 4n$	56

Figure	Page
2.10 Wavefunction Renormalization Parameters $f_{n,s}$ as Functions of Landau Level index for Four Different Type Filling Factors of QH States at $T = 0.03\epsilon_l$	57
2.11 Chemical Potential Differences $(\mu_{n,s} - \mu_s)$ as Functions of Landau Level index for Four Different Type Filling Factors of QH States at $T = 0.03\epsilon_l$	58
2.12 Singlet Haldane Masses $\Delta_{n,s}$ as Functions of Landau Level index for Four Different Type Filling Factors of QH States at $T = 0.03\epsilon_l$	59
2.13 Triplet Chemical Potentials $\tilde{\mu}_{n,s}$ and Dirac Masses $\tilde{\Delta}_{n,s}$ as Functions of Landau Level index for Different Type Filling Factors of QH States at $T = 0.03\epsilon_l$	60
2.14 Wavefunction Renormalization Parameters $f_{n,s}$ as Functions of Landau Level index for Different Type Filling Factors of QH States at $T = 0.08\epsilon_l$	61
2.15 Chemical Potential Differences $(\mu_{n,s} - \mu_s)$ as Functions of Landau Level index for Four Different Type Filling Factors of QH States at $T = 0.08\epsilon_l$	61
2.16 Singlet Haldane Masses $\Delta_{n,s}$ as Functions of Landau Level index for Four Different Type Filling Factors of QH States at $T = 0.08\epsilon_l$	62
3.1 Sampling of the Integrand of Δ_n as a Function of y_3 with all Other Variables at Fixed Values. Integrands with Sampling Functions, Gaussian, 4th-order Power-law, Cauchy, and Gamma Applied are Compared with the Original Integrand. .	76
3.2 Chiral Shift Δ_n as a Function of Longitudinal Momentum p_3 for $n = 1, 2, 3, 4$ Landau Levels and Chiral Shift Δ_n at Fermi Surface.	77
3.3 Chiral Chemical Potential $\mu_{5,n}$ as a Function of Longitudinal Momentum p_3 for $n = 1$ Landau Level and Chiral Chemical Potential $\mu_{5,n}$ at Fermi Surface. . .	79
3.4 Chiral Asymmetry $(p^3 - p_0^3)$ of Fermi Surface for Predominantly Left-handed and Right-handed Particles as a Function of Landau Level index.	80

CHAPTER 1

INTRODUCTION

Magnetic fields exist universally in stars, galaxies and the early Universe and can be applied externally to physical systems to explore their physical properties. Many effects of external magnetic fields in relativistic systems, such as the pulsar kicks due to the chiral asymmetry of fermions in magnetized stellar matter [1, 2, 3], the symmetry breaking phase transition in the early Universe [4, 5, 6], the chiral magnetic effect in the quark-gluon plasma produced in heavy-ion collisions [7, 8, 9, 10, 11], the quantum Hall effect in graphene [12, 13, 14, 15, 16, 17] and the anomalous magnetotransport in 3+1 dimensional Dirac/Weyl semimetals [18, 19, 20], were studied extensively using relativistic quantum field theory.

One of the important effects of magnetic fields on the dynamics of (quasi-)relativistic systems is the so-called magnetic catalysis [21]. The essence of the effect is the enhancement of dynamical chiral symmetry breaking due to fermion-antifermion pairing and the generation of a dynamical fermion mass (energy gap) [22, 23]. The underlying physics of magnetic catalysis lies in the dimensional reduction of $D \rightarrow D - 2$ in the low-energy dynamics of Dirac quasiparticles in an external magnetic field [21, 24, 25], which is due to the fact that the motion of charged particles are restricted to the directions perpendicular to the magnetic field. And the chiral condensate at weak coupling comes mostly from the fermions in the lowest Landau level with $D - 2$ dimensional dynamics.

The effects of magnetic fields in catalyzing spontaneous symmetry breaking were studied in the Nambu-Jona-Lasinio (NJL) model with local four-fermion interactions in 3+1 dimensions [22, 23, 25, 21, 24, 26] and the Gross-Neveu model in 2+1 dimensions [27, 28, 29], and in gauge theories with long-range Coulomb interactions such as QED and QCD [30, 31, 32, 33]. While it was known that in absence of a magnetic field, the spontaneous symmetry breaking and the generation of a dynamical mass appear only at coupling

constant greater than a critical value $g > g_c$ [34, 35, 36], the studies in [27, 28, 29] with a magnetic field showed that the symmetry breaking and the generation of a dynamical mass occur at any nonzero coupling $g > 0$, as long as there is an attractive fermion-antifermion interaction.

The review of the NJL model in [37] pointed out the different roles of magnetic fields in the NJL model and in the Bardeen-Cooper-Schrieffer (BCS) theory for low-temperature superconductivity. A constant magnetic field enhances fermion-antifermion pairing and symmetry breaking in the NJL model while tends to break superconductivity and restore symmetry in superconductors, where the opposite alignments of magnetic moments in Cooper pairs is energetically disfavored by the interaction with the magnetic field. Nevertheless, the role of the magnetic field in the dynamics of relativistic fermions in 2+1 dimensions has some similarities to that of the Fermi surface in the Bardeen-Cooper-Schrieffer (BCS) theory of low-temperature superconductivity [38, 39]. It strongly enhances fermion-antifermion pairing at low energies and is, therefore, responsible for the generation of a dynamical fermion mass.

We will first review the physics of magnetic catalysis in Section 1.1. The applications to graphene quantum Hall effect and chiral asymmetry in cold dense QED plasma will be introduced in Sections 1.2 and 1.3, respectively.

1.1 Magnetic Catalysis

1.1.1 Magnetic Catalysis in 2+1 Dimensions

The dynamics of free Dirac fermions in a magnetic field in 2+1 dimensions was studied in [26]. By introducing a nonzero fermion mass m in the theory and taking the zero mass limit $m \rightarrow 0$ in the solution, it was shown that the flavor symmetry breaking is already present in the free theory.

The Lagrangian density for noninteracting relativistic fermions takes the form

$$\mathcal{L} = \bar{\Psi}(t, \mathbf{r})(i\gamma^\mu D_\mu - m)\Psi(t, \mathbf{r}), \quad (1.1)$$

where $\mathbf{r} = (x, y)$ is the 2-dimensional position vector. The covariant derivative is given by $D_\mu = \partial_\mu + ieA_\mu^{ext}$ with electric charge $e < 0$, and the vector potential in the Landau gauge reads $A_\mu^{ext} = (0, -\mathbf{A}^{ext}) = (0, By, 0)$. To preserve the parity of the theory, a reducible four-dimensional representation of the gamma matrices [40, 41] was used

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad (1.2)$$

$$\gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad \gamma^3 = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (1.3)$$

In the massless limit $m \rightarrow 0$, the Lagrangian density is invariant under the global $U(2)$ flavor symmetry with the generators given by

$$T_0 = I, \quad T_1 = \gamma^5, \quad T_2 = \frac{1}{i}\gamma^3, \quad T_3 = \gamma^3\gamma^5. \quad (1.4)$$

The mass term breaks the $U(2)$ symmetry down to the $U(1) \times U(1)$ subgroup with the generators T_0 and T_3 . The energy spectrum was obtained by solving the Dirac equation [42, 26]

$$\begin{aligned} E_0 &= \pm m, & n &= 0, \\ E_n &= \pm \sqrt{m^2 + 2n|eB|}, & n &\geq 1. \end{aligned}$$

The density of states of the lowest Landau level is calculated to be $|eB|/2\pi$, which is half of that of the higher Landau levels $|eB|/\pi$. At $m \ll \sqrt{|eB|}$, the low-energy sector is determined by the lowest Landau level states with $E_0 = \pm m$.

The flavor condensate was derived in the Landau level representation and was found to come exclusively from the lowest Landau level in the massless limit,

$$\langle 0 | \bar{\Psi} \Psi | 0 \rangle \equiv - \lim_{x' \rightarrow x} \text{tr}[S(x, x')] \simeq - \frac{|eB|}{2\pi} \text{sign}(m), \quad (1.5)$$

where $S(x, x')$ is the free fermion propagator. Notably, in the massless limit $m \rightarrow 0$, the condensate equals to the area density of the lowest Landau level states. Since it is nonzero, there is already spontaneous $U(2)$ chiral symmetry breaking in Dirac fermions in 2+1 dimensions in a magnetic field even without interaction between fermions. The physical origin of this phenomenon is the infinite degeneracy of the vacuum state and the one-dimensional character of the dynamics of the lowest Landau level due to the dimensional reduction.

In the NJL model for relativistic fermions with four-fermion interactions in 2+1 dimensions and at vanishing magnetic field $B \rightarrow 0$, the dynamical mass is generated only at $g > g_c$ [36]. At magnetic field $B \neq 0$, the fermion condensate (1.5) is nonzero and the dynamical mass is generated even at weak coupling $g \ll g_c$. The mass increases with magnetic field B at any coupling constant g .

The study of the effective potential both numerically and analytically in the NJL model at nonzero temperature and chemical potential to the leading order of $1/N$ [26, 43, 44, 45, 46, 47, 48] shows a second order symmetry restoring phase transition with the dynamical mass vanishing above some critical temperature $T \geq T_c$. The critical temperature increases with the increase of the coupling constant g and magnetic field B . According to the Mermin-Wagner-Coleman theorem [49, 50], in 2+1 dimensions at nonzero temperature, the continuous symmetry cannot be spontaneously broken because strong fluctuations destroy the order parameter. Therefore, the corresponding phase transition, in fact, should be a Berezinsky-Losterliz-Thouless transition [51, 52].

The chemical potential affects the behavior of the potential and the dynamical fermion mass only when it is larger than the dynamical fermion mass $|\mu| > m_{dyn}$. The symmetry restoring critical chemical potential μ_c increases with the coupling constant g and magnetic field B in most regions except at weak field and $g > g_c$ [36]. The application of magnetic catalysis in 2+1 dimensions to the quantum Hall effect in graphene will be

discussed in more details in Section 1.2.

1.1.2 Magnetic Catalysis in 3+1 Dimensions

The analysis of the magnetic catalysis in 3+1 dimensions is very similar to that in 2+1 dimensions, although there are important differences in results. In the 3+1 dimensional NJL model in a magnetic field with local four-fermion interactions, the dynamical fermion mass breaks the $U(1)_L \times U(1)_R$ chiral symmetry to $U(1)_{L+R}$ subgroup. At weak coupling $g \ll g_c$ the dynamical mass is nonzero and shows an essential singularity at $g = 0$. Hence perturbative methods are not applicable for the dynamics of fermions in 3+1 dimensions in a magnetic field. Here we briefly review magnetic catalysis in the NJL model with four-fermion interaction in 3+1 dimensions.

With the magnetic field in the z -direction described by the following vector potential in the Landau gauge $A_\mu^{ext} = (0, -\mathbf{A}^{ext}) = (0, By, 0, 0)$, the energy spectrum was obtained in [42]

$$E_n(k_z) = \pm \sqrt{m^2 + 2n|eB| + (k_z)^2}, \quad (1.6)$$

where $n = k + s + \frac{1}{2} = 0, 1, 2, \dots$ is the Landau level index, k is the orbital motion quantum number and $s = \pm \frac{1}{2}$ is the spin quantum number. The lowest Landau level $n = 0$ corresponds to the lowest orbital state $k = 0$ and the spin projection $s = -\frac{1}{2}$. Hence, the lowest Landau level is spin polarized and has half the degeneracy of the higher Landau levels $n \geq 1$. The state area density in the perpendicular plane at a fixed longitudinal momentum k_z is same as that in the 2+1 dimensions, $|eB|/2\pi$ for $n = 0$ Landau level and $|eB|/\pi$ for higher Landau levels $n > 0$.

For a Dirac mass much smaller than the magnetic energy scale $m \ll \sqrt{|eB|}$, the low-energy excitations are determined by the lowest Landau level ($n = 0$) with $E(k_z) = \pm \sqrt{m^2 + k_z^2}$, which is the same as the spectrum of fermions in 1+1 dimensional quantum field theory. The dimensional reduction of 3+1 $\rightarrow$ 1+1 is the result of the partially restricted

motion of Dirac particles in the plane perpendicular to the magnetic field. In the mean-field approximation and weak coupling approximation, the calculation of the chiral condensate through fermion propagator shows that $\langle 0 | \bar{\Psi} \Psi | 0 \rangle \sim |eB| m \ln m$, i.e. it vanishes in the limit of zero bare mass $m \rightarrow 0$. Hence there is no spontaneous chiral symmetry breaking for free fermions in 3+1 dimensions in a magnetic field. The symmetry breaking is restored at $T \geq T_c$ and $\mu \geq \mu_c$.

1.1.3 Magnetic Catalysis in QED

The magnetic catalysis in QED which includes the photon field mediating the long-range interactions was analyzed in [26, 53, 54] from the Bethe-Salpeter equation for the NG modes in the ladder approximation and improved ladder approximation at weak coupling. At nonzero magnetic fields the chiral symmetry at the Lagrangian density $SU(N_f)_L \times SU(N_f)_R$ breaks down to $SU(N_f)_{L+R}$ subgroup with the generation of a dynamical mass m_{dyn} and $N_f^2 - 1$ NG gapless bosons [36].

When the magnetic field is the largest scale, the contribution to the photon polarization operator is of the order of $\alpha |eB|$ and leads to a strong screening of the interaction. The screening effects in QED can be considered in the improved ladder approximation by replacing the free photon propagator with the one-loop expression. The analysis in [55] with screening taken into account gave the result for the dynamical mass similar to that without screening, but with the change of $\alpha \rightarrow \alpha/2$. Due to the smallness of the dynamical fermion mass $m_{dyn} \ll \sqrt{|eB|}$, there are large logarithmic contributions or mass singularities of the type $\ln(|eB|/m_{dyn}^2) \sim \alpha^{-1/2} \gg 1$ in the infrared region. This leads to breaking of the diagrammatic expansion of the Bethe-Salpeter equation in powers of α . This was resolved by using a special gauge choice in the Schwinger-Dyson equation, in which the higher order corrections are suppressed [53, 54].

The relativistic field models are effective in describing the quasiparticle excita-

tions in planar condensed matter systems such as high- T_c superconductors and graphene. With gauge fields describing interparticle interactions propagating in the 3-dimensional space and fermion fields describing electron and hole quasiparticles propagating on the 2-dimensional plane, the reduced 3+1 dimensional gauge theories [56, 57, 58, 59, 60] is used for the study of the dynamics of 2-dimensional massless Dirac fermions in graphene. The results showed that, at energy scale below the Landau scale $\omega < \varepsilon_B$, the pairing dynamics is dominated by the lowest Landau level and the infrared region $\omega \lesssim m_{dyn}$. At near critical coupling $g \sim g_c$ and strong coupling $g > g_c$, all Landau levels contribute to the pairing dynamics and the dynamical mass is of the order of the gap $\Delta_0 \sim \varepsilon_B$, which is in qualitative agreement with the results studied in a (2+1)-dimensional NJL model [26, 55].

1.2 Quantum Hall Effect in Graphene

Graphene is a two dimensional hexagonal lattice made of carbon atoms. Graphene was first theoretically studied in [61] and was presumed not to exist in a free state because of a thermodynamical instability [62, 63]. It was discovered by a micromechanical cleavage (repeated peeling) of bulk graphite in the laboratory in 2004 [64, 65]. The structure of graphene lattice is shown in Fig. 1.1 with two atoms A and B per unit cell. The lattice vectors are $\mathbf{a}_1 = \frac{a}{2}(3, \sqrt{3})$ and $\mathbf{a}_2 = \frac{a}{2}(3, -\sqrt{3})$, respectively, where $a \approx 1.42 \text{ \AA}$ is the distance between adjacent atoms. The reciprocal-lattice vectors are $\mathbf{b}_1 = \frac{2\pi}{3a}(1, \sqrt{3})$, $\mathbf{b}_2 = \frac{2\pi}{3a}(1, -\sqrt{3})$, and the two Dirac points in the Brillouin zone in momentum space are $\mathbf{K}_1 = (\frac{2\pi}{3a}, \frac{2\pi}{3\sqrt{3}a})$, and $\mathbf{K}'_2 = (\frac{2\pi}{3a}, -\frac{2\pi}{3\sqrt{3}a})$, respectively.

1.2.1 Dirac Fermions in Strong Magnetic Field

In the tight-binding approximation that takes into account the hopping of electrons to the nearest- and next-nearest-neighbor atoms, the electron Hamiltonian of graphene takes the

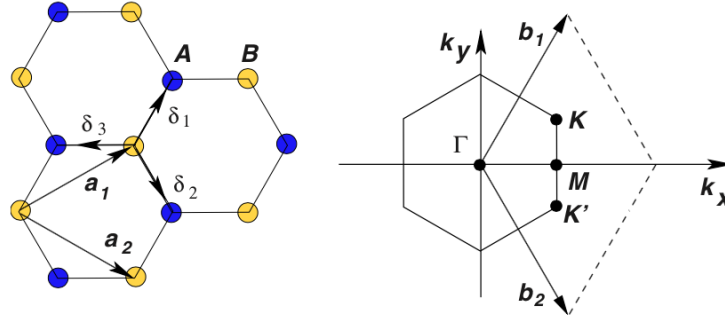


Figure 1.1: Taken from Fig. 2 in [66]. Left: The lattice structure of graphene in the coordinate space with sublattices A and B and lattice unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Right: The Brillouin zone in the reciprocal space with two valleys of momentum $\mathbf{K}$ and $\mathbf{K}'$ and reciprocal-lattice vectors $\mathbf{b}_1$ and $\mathbf{b}_2$.

form

$$H = -t \sum_{\langle i,j \rangle, \sigma} a_{\sigma,i}^\dagger b_{\sigma,j} - t' \sum_{\langle\langle i,j \rangle\rangle, \sigma} (a_{\sigma,i}^\dagger a_{\sigma,j} + b_{\sigma,i}^\dagger b_{\sigma,j}) + H.c., \quad (1.7)$$

where the first term represents the annihilation and the creation of an electron of spin σ ($\sigma = \uparrow, \downarrow$) on the three nearest-neighbor sites on sublattices A and B , and the second term represents that on six next nearest-neighbor sites on sublattices A and B . Parameters t and t' are the corresponding hopping energies between different sublattices and within the same sublattice, respectively. The energy bands derived from the Hamiltonian are given by [61]

$$E_{\pm}(\mathbf{k}) = \pm t \sqrt{3 + f(\mathbf{k})} - t' f(\mathbf{k}), \quad (1.8)$$

where $f(\mathbf{k}) = 2 \cos(\sqrt{3} k_y a) + 4 \cos(\frac{\sqrt{3}}{2} k_y a) \cos(\frac{3}{2} k_x a)$. For $t' = 0$ the spectrum is symmetric around zero energy, and for $t' \neq 0$ the electron-hole symmetry is broken. The band structure of graphene is shown in Fig. 1.2. Expanding the band structure around the Dirac points $\mathbf{K}$ (or $\mathbf{K}'$) with $\mathbf{k} = \mathbf{K} + \mathbf{q}$ and $|\mathbf{q}| \ll |\mathbf{K}|$, a linear energy dispersion relation is obtained [61]

$$E_{\pm}(\mathbf{q}) \approx 3t' \pm v_F |\mathbf{q}| + O[(q/K)^2], \quad (1.9)$$

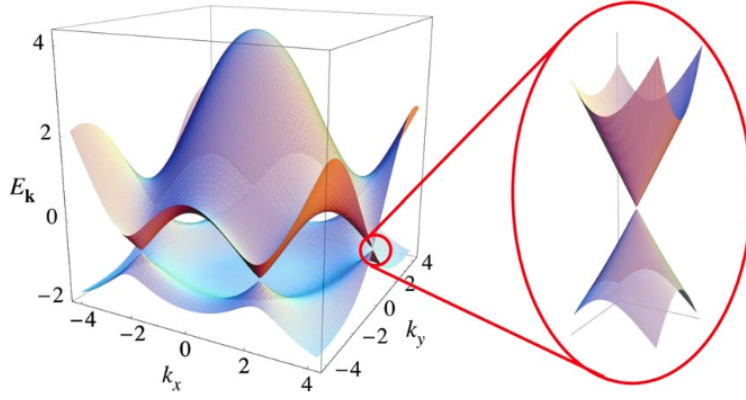


Figure 1.2: Taken from Fig. 3 in [66]. Left: The band structure of graphene electrons at Dirac points. Right: Zoom in of the energy band at one Dirac point.

where $v_F = 3ta/2 \approx 1 \times 10^6 m/s$ is the Fermi velocity, which is about 300 times smaller than the speed of light.

Therefore the low-energy excitations in graphene are analogous to massless relativistic particles with a linear dispersion relation. The valence and conduction bands meet at two Dirac points (labeled K and K') at the corners of the Brillouin zone. Hence, graphene is a two-dimensional zero-bandgap semimetal with Dirac fermions with the speed $v_F \approx 10^6 m/s$. At low energies, the effective Hamiltonian takes the form [67]

$$H = -iv_F \int dxdy \left[\hat{\Psi}_1^\dagger(\mathbf{r}) \boldsymbol{\sigma} \cdot \nabla \hat{\Psi}_1(\mathbf{r}) + \hat{\Psi}_2^\dagger(\mathbf{r}) \boldsymbol{\sigma}^* \cdot \nabla \hat{\Psi}_2(\mathbf{r}) \right], \quad (1.10)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$, and spinors $\hat{\Psi}_i^\dagger = (a_i^\dagger, b_i^\dagger)$ are made of two-component wave functions in momentum space around $\mathbf{K}$ and $\mathbf{K}'$ points, i.e.,

$$\psi_{\pm, \mathbf{K}}(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta_{\mathbf{k}}/2} \\ \pm e^{i\theta_{\mathbf{k}}/2} \end{pmatrix}, \quad \psi_{\pm, \mathbf{K}'}(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta_{\mathbf{k}}/2} \\ \pm e^{-i\theta_{\mathbf{k}}/2} \end{pmatrix}. \quad (1.11)$$

The corresponding spinors obey the 2D Dirac equation

$$-iv_F \boldsymbol{\sigma} \cdot \nabla \psi(\mathbf{r}) = E \psi(\mathbf{r}) \quad (1.12)$$

with eigenenergies $E = \pm v_F k$. The two wave functions are related by time-reversal symmetry. The eigenstates have a well-defined helicity, which is the projection of the momentum operator $\mathbf{p}$ on the (pseudo)spin $\boldsymbol{\sigma}$, namely $\hat{h} = \frac{1}{2} \boldsymbol{\sigma} \cdot \hat{\mathbf{p}}$. The helicity values $\pm 1/2$ are good quantum numbers close to the Dirac points K and K' . At higher energies or $t' \neq 0$, the electron-hole symmetry is broken and helicity stops to be a good quantum number.

In a uniform magnetic field B perpendicular to the graphene plane, the cyclotron motion of Dirac fermions has the frequency $\omega_c = \sqrt{2} \frac{v_F}{l_B}$, where $l_B = \sqrt{\frac{c}{eB}}$ is the magnetic length. The corresponding cyclotron energy is about 1000 K at $B \approx 10$ T. This is two orders of magnitude larger than that of the usual nonrelativistic 2D electron gas, and thus enables the observation of the quantum Hall effect even at room temperature [68]. In the Landau gauge $\mathbf{A} = (-By, 0)$, the Dirac equation takes the form

$$v_F [\boldsymbol{\sigma} \cdot (-i\nabla + e\mathbf{A}/c)] \psi(\mathbf{r}) = E \psi(\mathbf{r}), \quad (1.13)$$

and the wave function has the form $\psi(x, y) = e^{ikx} \phi(y)$. Using Landau level representation, the Landau level energies of the massless relativistic fermions in quantized fields can be derived in the same way as in [69],

$$E_n = \pm \sqrt{2e\hbar v_F^2 B \left| n + \frac{1}{2} \pm \frac{1}{2} \right|}, \quad (1.14)$$

where $n = 0, 1, 2, \dots$ is the Landau level index, and the ‘ $\pm$ ’ sign under the square root corresponds to the two ‘pseudospins’ that originate from the sublattice degeneracy. As one can see, the degenerate states at $n = 0$ are allowed only for pseudospin ‘ $-$ ’. Therefore, the degeneracy of the lowest Landau level is half of the four-fold degeneracy at all other Landau levels $n \geq 1$. Half of these states are hole states and the other half are electron states.

1.2.2 Graphene QHE: Experiments

The experimental study of the electric field effect in graphene [65] shows that the graphene's conductivity σ is proportional to the gate voltage V_g with positive (negative) gate voltages inducing electron (hole) charge carriers. The concentration of charge carriers n is linearly proportional to the gate voltage V_g . High mobilities of charge carriers were found from the linear dependence $\sigma(V_g)$ with $\mu = \sigma/ne$ reaching $15,000 \text{ cm}^2\text{V}^{-1}\text{s}^{-1}$ independent of temperature T between 10 and 100 K.

The Shubnikov-de Haas oscillations (SdHOs) of graphene electrons and holes show same fundamental dependence on the magnetic fields B , the gate voltages V_g and temperatures T . The fundamental SdHO frequency exhibits linear dependence on the carrier concentration $B_F = \beta n$ and yields degeneracy $f = 4$ for the 2D system, which agrees with the spin-sublattice degeneracy in graphene. The dependence of the SdHO amplitude on temperature T are fitted by $T / \sinh(2\pi^2 k_B T m_c / \hbar e B)$, where $m_c = E / v_F^2$ is the cyclotron mass with values between ~ 0.02 - $0.07 m_0$ (the free electron mass) [65]. The experimental dependence of m_c on the carrier concentration $m_c \propto n^{1/2}$ combined with the linear dependence of B_F yield the linear dispersion relation $E = \hbar k v_F$ with $v_F \approx 10^6 \text{ m/s}$. Hence, this reconfirms that the charge carriers in graphene can be indeed approximated by massless relativistic particles, whose wave functions satisfy the 2-dimensional Dirac equation.

In the magnetic field B less than about 20 T, the Hall conductivity $\sigma_{xy} \equiv v \frac{e^2}{h}$ as a function of the electron and hole concentrations shows the plateaus with values $\nu = \pm 4(n + 1/2)$ shown in Fig. 1.3 [65]. The prefactor 4 reflects the four-fold spin and sublattice-valley degeneracy in the graphene band structure. The shift by one half in the filling factor expression is related to the relativistic properties of massless Dirac fermions and the completely spin-polarized nature of the lowest Landau level $n = 0$ state.

The authors of [70] studied the quantum Hall effect of high quality suspended

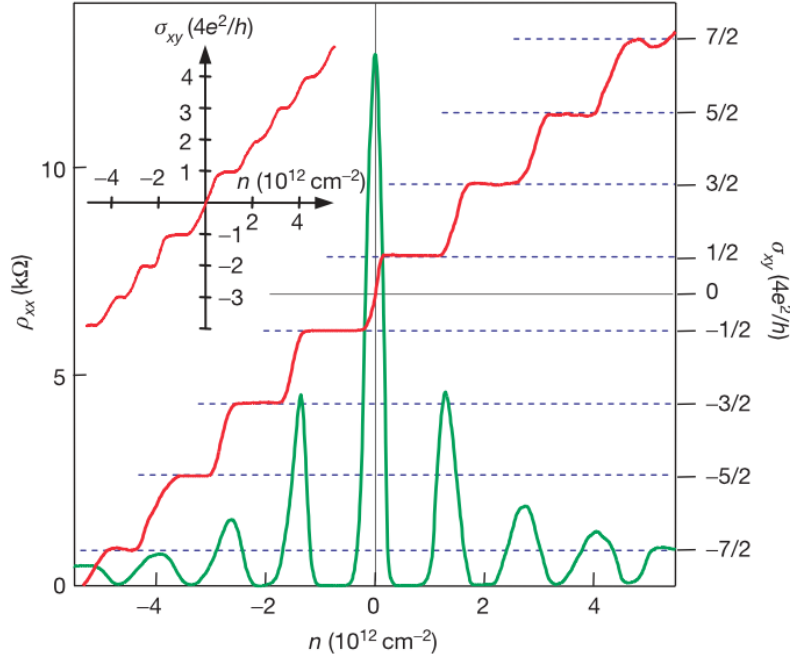


Figure 1.3: Taken from Fig. 4 in [65]. Quantum Hall effect in graphene: the dependence of the Hall conductivity σ_{xy} on the carrier concentration shows plateaus at filling factors $\nu = 4(n + 1/2)$. Inset: The integer QHE in “two-layer graphite”.

graphene in strong magnetic field $B > 25$ T and found new QH plateaus at filling factors $\nu = 0, \pm 1, \pm 4$ other than the plateaus at $\nu = \pm 4(n + 1/2)$ observed in lower magnetic field $B < 9$ T (Fig. 1.4). The observation of plateaus at $\nu = 0, \pm 1$ implies the fully lifted fourfold spin-sublattice degeneracy of the $n = 0$ Landau level state at high magnetic fields. The plateaus at $\nu = \pm 4$ are possible if the fourfold degeneracy of the $n = \pm 1$ states is lifted at least partially. Examining the dependence of the energy gap ΔE on the tilting of the magnetic field, the $\nu = \pm 4$ energy gaps show linear dependence on the total magnetic field irrespective of the tilt angle, suggesting that the spin degeneracy of the $n = \pm 1$ Landau level states is lifted. As the sublattice symmetry is not broken by the magnetic field, the lifted degeneracy of the $n = 0$ Landau level states involve the many-body electron interactions and the formation of a charge density wave due to Fermi surface nesting of the electron and hole

LLs. The $\nu = \pm 1$ energy gaps show a $\sqrt{B}$ dependence on the perpendicular component of the magnetic field [71] suggesting a nonspin origin and a sublattice symmetry breaking driven by the electron-electron interactions [72, 73, 74, 75, 76, 77, 78, 79, 14, 80, 81, 82]. Other experiments also observed plateaus at filling factors $\nu = 1/3, 1, 3$ [83] and fractional filling factors $\nu = 1/3, 1/2, 2/3$ [84].

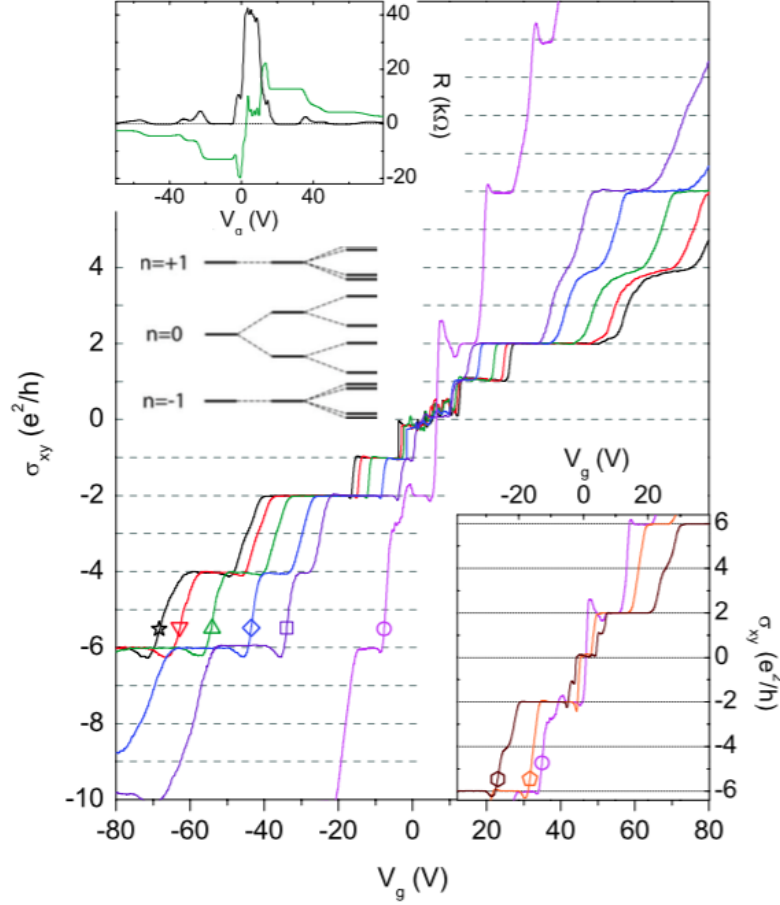


Figure 1.4: Taken from Fig. 2 in [70]. Quantum Hall effect in graphene: dependence on the applied voltage V_g at different magnetic field (9 T circle, 25 T square, 30 T diamond, 37 T up triangle, 42 T down triangle, and 45 T star) of the Hall conductivity σ_{xy} . New plateaus at filling factors $\nu = 0, \pm 1, \pm 4$ appear at magnetic fields $B > 25$ T other than the standard plateaus at $\nu = \pm 4(n + 1/2)$.

1.2.3 Graphene QHE: Theory

The theoretical explanations of the observed quantum Hall plateaus include two scenarios. One is the quantum Hall ferromagnetism (QHF) with order parameters given by the spin and valley charge densities imbalance [72, 85]. Another one is the magnetic catalysis introduced in the previous section, which is characterized by order parameters given by the excitonic condensates and generates the Dirac masses [14, 86, 78, 79, 87, 80, 81]. The study in a model with local four-fermion interactions show that these two sets of order parameters may coexist and indicate same dynamical origin [12, 13, 16]. We introduce the studies of graphene QHE in the framework of quantum field theories with both QHF and MC order parameters. We will discuss models with short-range interactions and long-range Coulomb interactions in this section.

Quantum Hall effect was studied in [13] in the clean limit of suspended graphene, in which lifting of the Landau level degeneracy comes as a result of symmetry-breaking dynamics triggered by short-range point-like density-density interaction. The corresponding simplified model demonstrates the key features of the underlying physics.

In the low-energy effective theory, the free quasiparticle Hamiltonian is given by

$$H_0 = v_F \int d^2r \bar{\Psi}(\gamma^1 \pi_x + \gamma^2 \pi_y) \Psi, \quad (1.15)$$

where $\Psi_s^T = (\psi_{KAs}, \psi_{KBs}, \psi_{K'Bs}, -\psi_{K'As})$ is the four-component Dirac spinor with spin indices $s = \uparrow, \downarrow$, sublattice indices (A, B) and two inequivalent valley points in momentum space (K, K') . The canonical momentum $\pi = -i\hbar\nabla - e\mathbf{A}/c$ includes the vector potential $\mathbf{A}$ of the magnetic field component $\mathbf{B}_\perp$ orthogonal to the graphene plane.

The Hamiltonian of the short-range interaction takes the form

$$H_C = \frac{1}{2} \int d^2\mathbf{r} d^2\mathbf{r}' \Psi^\dagger(\mathbf{r}) \Psi(\mathbf{r}) U_C(\mathbf{r} - \mathbf{r}') \Psi^\dagger(\mathbf{r}') \Psi(\mathbf{r}'), \quad (1.16)$$

where the point-like contact interaction $U_C = G_{int} \delta^2(\mathbf{r})$ and G_{int} is a dimensionful coupling constant. The contact interaction preserves the global $U(4)$ symmetry in the Hamiltonian. The electron chemical potential and the Zeeman interaction terms are included in the Hamiltonian as

$$H = H_0 + H_c + \int d^2\mathbf{r} (\mu_B B \Psi^\dagger \sigma^3 \Psi - \mu_e \Psi^\dagger \Psi). \quad (1.17)$$

The Zeeman term breaks the symmetry down to $U_\uparrow(2) \times U_\downarrow(2)$. Because of the renormalization, the strength of the low-energy coupling associated with the short-range interactions may become rather large and even more important than the Zeeman interaction. Moreover, there could exist many symmetry-breaking interaction terms that could be important in determining the underlying structure and the preferred alignment of the ground state alignment at different filling factors.

The magnetic catalysis and ferromagnetic order parameters are incorporated in the full quasiparticle propagator. The magnetic catalysis order parameters are Dirac mass $\tilde{\Delta}_s$ and Halden mass Δ_s which represents the electron density imbalances at two valleys and two sublattices (i.e. charge density wave CDW), respectively. The triplet Dirac mass term $\tilde{\Delta}_s \bar{\Psi} P_s \Psi$ is even under time reversal $\mathcal{T}$, the singlet Halden mass term $\Delta_s \bar{\Psi} \gamma^3 \gamma^5 P_s \Psi$ is odd under time reversal $\mathcal{T}$ [88]. The quantum Hall ferromagnetism order parameters, the chemical potentials μ_3 and $\tilde{\mu}_s$, are related to the spin density $\langle \Psi^\dagger \sigma^3 \Psi \rangle$ and the anomalous magnetic pseudomoment $\langle \Psi^\dagger \gamma^3 \gamma^5 P_s \Psi \rangle$.

The analysis of the solutions of the gap equations showed that the MC and QHF order parameters should always coexist, and the order parameters on the LLL are not independent [13]. The analytical solutions at zero temperature and the numerical solutions at nonzero temperature of the gap equations produce singlet and hybrid solutions for various filling factors of the Landau level $n = 0$ and $n = 1$. The Zeeman splitting symmetry breaking of the half-filled states is strongly enhanced by the dynamical contribution. For the hybrid

solutions with nonvanishing both singlet and triplet order parameters corresponding to the quarter and three-quarter filled states, the flavor symmetry $U_\uparrow(2) \times U_\downarrow(2)$ breaks down to $U_\uparrow(1) \times U_\uparrow(1) \times U_\downarrow(2)$ and $U_\uparrow(2) \times U_\downarrow(1) \times U_\downarrow(1)$. With the increase of the temperature, the hybrid solutions gradually vanish and the broken symmetry is restored.

The effects of the long-range Coulomb interaction in the graphene QHE was studied in [89]. In the Landau level representation for the fermion propagator, it is shown that the long-range interaction induces the running of the dynamical parameters with the Landau level index n and reproduce the quantum Hall states at any filling factors. The model with the long-range Coulomb interaction reveals a phase structure similar to that with the short-range interaction, but quantitatively different.

1.3 Chiral Asymmetry in a Magnetic Field

The anomalous symmetries are the symmetries that are absent in the full quantum field theory while present in the classical Lagrangian. Examples of anomalous symmetries are behind the apparent contradiction between the measured neutral pion decay rate $\Gamma(\pi^0 \rightarrow \gamma\gamma)$ and a theoretical prediction assuming the axial $U(1)_A$ symmetry in the field equations of motion [90, 91, 92], and the much larger mass of η' meson compared to the other eight pseudoscalar mesons from $SU(3)_A$ chiral symmetry breaking [93, 94]. The anomalous $U(1)_A$ chiral symmetry is equivalent to a nonvanishing divergence of the axial current. In the massless QED, the anomaly is expressed as [91, 92]

$$\partial_\mu j_5^\mu = -\frac{e^2}{16\pi^2} \epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda} F_{\mu\nu}, \quad (1.18)$$

where $j_5^\mu = \bar{\psi} \gamma_5 \gamma^\mu \psi$ is the axial current density and the anomaly relation denotes the non-conserved number densities of left-handed and right-handed fermions. There are two interesting effects, the chiral separation effect (CSE) [95, 96, 97, 98, 99, 100, 101] and the chiral magnetic effect (CME) [102, 103, 104, 105], related to the anomalous chiral properties of relativistic matter at nonzero density in an external magnetic field.

1.3.1 Chiral Separation and Chiral Magnetic Effects

The analysis in [42] for free fermions in 3+1 dimensions in a magnetic field showed that the lowest Landau level is completely spin-polarized and each state has a definite chirality in the case of massless fermions. At a nonzero chemical potential μ , the axial current density is determined by the total number density of the occupied LLL states as [95, 96]

$$\mathbf{j}_5 = \frac{e\mathbf{B}}{2\pi^2}\mu. \quad (1.19)$$

This is the so-called chiral separation effect. The chiral separation effect appears in free theory and only comes from the contribution of the lowest Landau level [96], which is the same as that of the chiral anomaly [106]. The chiral magnetic effect (CME) refers to a nondissipative electric current $\mathbf{j}$ induced by the chiral chemical potential μ_5 , namely

$$\mathbf{j} = \frac{e\mathbf{B}}{2\pi^2}\mu_5. \quad (1.20)$$

The combination of the chiral separation and chiral magnetic effects gives rise to the chiral magnetic wave (CMW) [107], a propagating perturbation of alternating electric and chiral charge density fluctuations. The observational study of the quadrupole correlations in heavy-ion collisions provides a test of the CMW [108, 109].

The chiral asymmetry present only in the lowest Landau level in free theory is promoted to all Landau levels in interacting theory, which was studied in both the NJL model [99, 110, 111] and QED [101, 112]. The asymmetry is quantified by the dynamically generated chiral shift parameter Δ , which describes the relative shift of the longitudinal momenta for opposite chirality fermions. Besides the chiral shift in all Landau levels, interaction between fermions also induces a nonzero radiative correction to the axial current.

1.3.2 Chiral Asymmetry in a Weak Magnetic Field

The study of chiral asymmetry in the NJL model has the advantage of the simplicity in capturing all the qualitative features of the chiral separation effect and illustrating the dy-

namics of the generation of the chiral shift Δ . It should be noted, however, that the NJL model is nonrenormalizable and, thus, the results of the chiral shift the axial current should be interpreted with caution.

The Lagrangian density of a single flavor fermion in a magnetic field in 3+1 dimensional NJL model reads

$$\mathcal{L} = \bar{\psi}(iD_\nu + \mu_0\delta_\nu^0)\gamma^\nu\psi - m_0\bar{\psi}\psi + \frac{G_{int}}{2} \left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2 \right], \quad (1.21)$$

where m_0 is the bare fermion mass and μ_0 is the chemical potential, G_{int} is the dimensionful coupling constant. The magnetic field is in the z -direction and the Landau gauge $A = (0, -By, 0, 0)$ is used. The only unbroken symmetry is the parity $\mathcal{P}$. The model possesses the $U(1)_L \times U(1)_R$ chiral symmetry in the chiral limit $m_0 = 0$, and the chiral symmetry is spontaneously broken in the vacuum state ($\mu_0 = 0$) and restored at large chemical potential when a nonzero chiral shift Δ generated in the normal ground state.

In the mean-field approximation, by assuming weak coupling $G_{int} \rightarrow 0$ and neglecting the wavefunction renormalization, with local contact interaction, the full fermion propagator in coordinate space takes the form [99],

$$iG^{-1}(u, u') = \left[(i\partial_t + \mu)\gamma^0 - (\boldsymbol{\pi} \cdot \boldsymbol{\gamma}) + i\tilde{\mu}\gamma^1\gamma^2 + \Delta\gamma^3\gamma^5 - m \right] \delta^4(u - u'), \quad (1.22)$$

where $u = (t, \mathbf{r})$ and the dynamical parameters $\tilde{\mu}$ and Δ represent anomalous magnetic moment and chiral shift parameter, respectively. In 2+1 dimensions, the analogous Δ term $\Delta\gamma^3\gamma^5$ is the Haldane mass that breaks the time-reversal symmetry and plays an important role in the quantum Hall effect in graphene. The parameters m and μ are the dynamical mass (Dirac mass) and effective chemical potential in interacting theory. The chiral shift was derived from the Schwinger-Dyson (gap) equation and takes the form

$$\Delta = -\frac{1}{2}G_{int}\langle j_5^3 \rangle, \quad (1.23)$$

where $\langle j_5^3 \rangle$ is the axial current density.

From the poles of the fermion propagator in the Landau-level representation, the dispersion relations show that the chiral shift is coupled to the longitudinal momenta with $k^3 \rightarrow k^3 \pm s_\perp \Delta$ of opposite chirality fermions in higher Landau levels, and becomes a part of the effective chemical potential $\mu - s_\perp \Delta$ in the lowest Landau level. This is because the dynamics of the LLL is 1+1 dimensional and the gauge transformation is anomalous [25, 113, 114].

At zero temperature and in the chiral limit with $m = m_0 = 0$ and $\langle \psi \bar{\psi} \rangle = 0$, to the zeroth order in coupling, the fermions are noninteracting and there is no chiral shift $\Delta = 0$ and $\mu = \mu_0$, while the fermion density $\langle j^0 \rangle$ and axial current density are nonzero with $\langle j_5^3 \rangle_0 = \frac{eB}{2\pi^2} \mu_0$. To the first order, the chiral shift

$$\Delta \simeq -\frac{1}{2} G_{int} \langle j_5^3 \rangle_0 = -G_{int} \frac{eB}{4\pi^2} \mu_0 \quad (1.24)$$

is nonzero and generated by perturbative dynamics. The chiral shift is insensitive to the temperature in cold dense matter such as in stars, while may be enhanced largely in hot matter such as heavy ion collisions at $T > \mu$. A nonzero chiral shift affects the axial current while the chiral anomaly relation is not changed [99, 91, 92].

The analytical analysis of the gap equations at zero temperature using gauge noninvariant regularization at momentum cutoff $|k^3| < \Lambda$, Landau level $n_{max} \equiv \frac{\Lambda^2}{2|eB|}$, and gauge invariant proper-time regularization, gives qualitatively same results. In the weak coupling limit $g \equiv \frac{G_{int} \Lambda^2}{4\pi^2} \ll g_{cr} = 1$, there are two types of solutions: a symmetry broken state at zero density with $m = 0$ and $\Delta \neq 0$, and a normal state with restored symmetry at finite density with $m \neq 0$ and $\Delta = 0$.

The numerical analysis for the gap equations was performed with cutoff regularization in longitudinal momentum and the Landau level at nonzero temperature. The results show that in the chiral limit $m_0 = 0$, the dynamical mass m gradually vanishes as the temperature increases. The effective chemical potential μ and the chiral shift Δ show similar

approximate linear behavior with the increase of μ_0 while are trivial below some critical μ_0 at different m_0 . At nonzero temperature, it shows that the dynamical mass m is suppressed as temperature increases and the chiral shift Δ is insensitive to temperature at $T \ll \mu_0$ (stellar matter) and increases with T at $T > \mu$ (heavy ion collisions).

With a nonzero chiral shift, the Fermi surface is defined from the dispersion relation as the hypersurface in longitudinal momentum k^3 and Landau level n at quasiparticle zero energy. In relativistic dense matter ($\mu \gg m$), the $k^3 \gg m$ for most Fermi surface except for the ones around n_{max} , the states are approximately same as the massless case and hence characterised by the chiralities, left-handed and right-handed. Assuming $\text{sign}(eB) < 0$, the Fermi surface for predominantly left-handed particles is

$$\begin{aligned} n = 0: \quad k^3 &= +\sqrt{(\mu + s_{\perp}\Delta)^2 - m^2}, \\ n > 0: \quad k^3 &= +\sqrt{(\sqrt{\mu^2 - 2n|eB|} - s_{\perp}\Delta)^2 - m^2}, \\ k^3 &= -\sqrt{(\sqrt{\mu^2 - 2n|eB|} + s_{\perp}\Delta)^2 - m^2}, \end{aligned} \quad (1.25)$$

and for predominantly right-handed particles is

$$\begin{aligned} n = 0: \quad k^3 &= -\sqrt{(\mu + s_{\perp}\Delta)^2 - m^2}, \\ n > 0: \quad k^3 &= -\sqrt{(\sqrt{\mu^2 - 2n|eB|} - s_{\perp}\Delta)^2 - m^2}, \\ k^3 &= +\sqrt{(\sqrt{\mu^2 - 2n|eB|} + s_{\perp}\Delta)^2 - m^2}. \end{aligned} \quad (1.26)$$

It is shown that the Fermi surface is asymmetric at every Landau level. From the equations of Fermi surface of a certain chirality fermions, the LLL and the higher Landau levels give opposite contributions to the overall chiral asymmetry. In case $\mu \gg \sqrt{|eB|}$ and many Landau levels are occupied, the overall asymmetry is dominated by higher Landau levels. In the opposite at superstrong magnetic field, only the LLL is occupied and the overall asymmetry is determined by the LLL.

The study of chiral asymmetry in the NJL model showed that a chiral shift Δ is generated in both free theory and interacting theory. As the NJL model is nonrenormalizable and the chiral anomaly relation is connected with the ultraviolet divergence, it is more reliable to study the radiative corrections to the axial current in the renormalizable QED gauge theories. The analysis of the ultraviolet divergency in the fermion self-energy shows that the dynamical functions are free of the UV divergency at $n \geq 1$ and it only effects the dynamical parameters at the lowest Landau level.

In a weak magnetic field and to the linear order in B , write the fermion propagator in free part of magnetic field and linear part of magnetic field, and the self-energy is split into the “vacuum” and “matter” parts

$$\bar{S}(k) = \bar{S}^{(0)}(k) + \bar{S}^{(1)}(k) + \dots, \quad (1.27)$$

$$\bar{\Sigma} = \bar{\Sigma}_{vac}^{(0)} + \bar{\Sigma}_{mat}^{(0)} + \bar{\Sigma}_{vac}^{(1)} + \bar{\Sigma}_{mat}^{(1)}. \quad (1.28)$$

The “vacuum” part is same as the vacuum self-energy in QED with substitution $p_0 \rightarrow p_0 + \mu$. The chiral asymmetrical parameters Δ and μ_5 are generated from the self-energy in the linear in magnetic field. Analyzing the propagator and self-energy analytically gives the chiral shift and the chiral chemical potential on the Fermi surface, i.e. at $p_0 \rightarrow 0$ and $|\mathbf{p}| \rightarrow p_F = \sqrt{\mu^2 - m^2}$

$$\Delta = \Delta_{mat} + \Delta_{vac} \simeq -\frac{\alpha e B \mu}{\pi m^2} \left(\ln \frac{m^2}{2\mu(|\mathbf{p}| - p_F)} - 1 \right), \quad (1.29)$$

$$\mu_5 = \mu_5^{mat} + \mu_5^{vac} \simeq -\frac{\alpha e B \mu p^3}{\pi m^2 p} \left(\ln \frac{m^2}{2\mu(|\mathbf{p}| - p_F)} - 1 \right). \quad (1.30)$$

The chiral asymmetry ($p^3 - p_0^3$) at the Fermi surface, the deviation of the longitudinal momentum from that without including interaction, were calculated numerically [100] and found that the Fermi surface of the predominantly left-handed particles and predominantly right-handed particles are shifted in the direction and the opposite direction of the

magnetic field. The chiral asymmetry, is qualitatively in agreement with the results in the NJL model, while it comes from contributions from both Δ and μ_5 , and dependent on the particles' momenta.

Chapter 2

GRAPHENE QUANTUM HALL EFFECT IN STATIC SCREENING

In this chapter, we study the effect of screening on the properties of the quantum Hall states with integer filling factors in graphene, by making use of the generalized Landau-level representation (GLLR) for the quasiparticle propagator. The analysis is performed in the low-energy Dirac model in the mean-field approximation, in which the long-range Coulomb interaction is modified by the one-loop static screening effects in the presence of a background magnetic field. By utilizing a rather general ansatz for the propagator, in which all dynamical parameters are running functions of the Landau-level index n , we derive a self-consistent set of the Schwinger-Dyson (gap) equations and solve them numerically. The explicit solutions demonstrate that static screening leads to a substantial suppression of the gap parameters in the quantum Hall states with a broken $U(4)$ flavor symmetry. The temperature dependence of the energy gaps is also studied. The corresponding results mimic well the temperature dependence of the activation energies measured in experiment. It is also argued that, in principle, the Landau-level running of the quasiparticle dynamical parameters could be measured via optical studies of the integer quantum Hall states.

2.1 Motivation

As predicted theoretically more than three decades ago [67, 115], the low-energy excitations in planar graphite, or graphene are described by $(2+1)$ -dimensional, massless Dirac fermions. Interestingly, the spinor structure of the corresponding fermion fields has nothing to do with the usual spin of electrons. It is connected with an effective “pseudospin” that has its roots in the hexagonal arrangement of carbon atoms in graphene, which can be viewed as a superposition of two inequivalent triangular sublattices. At the same time, the usual spin plays a rather passive role analogous to an extra “species” degree of freedom.

The Dirac nature of excitations was confirmed experimentally by the observation of the anomalous integer quantum Hall (QH) effect in Refs. [65, 116]. At sufficiently low temperatures, the measured Hall conductivity revealed a set of well-resolved QH plateaus at the filling factors $\nu = \pm 4(n + 1/2)$, where $n = 0, 1, 2, \dots$ is an integer. One of the signature properties of the corresponding QH effect is the anomalous shift of $1/2$ in the expression for the filling factor. This measurement appears to be in perfect agreement with the theoretical predictions of the low-energy Dirac theory [56, 117, 118, 119]. The overall factor of 4 in the expression for ν is also in agreement with the predicted fourfold (spin and sublattice-valley) degeneracy of the Landau levels in the low-energy theory. For reviews of quantum Hall physics in graphene see Ref. [66, 120, 121].

The spin and sublattice-valley degeneracy of the Landau levels in the low-energy theory of graphene can be associated with an approximate global “flavor” $U(4)$ symmetry. Strictly speaking, the symmetry is not exact. There exist several small explicit symmetry breaking effects that lift the degeneracy of the Landau levels [122]. The most obvious among them is the Zeeman effect that breaks the $U(4)$ symmetry down to $U_{\uparrow}(2) \times U_{\downarrow}(2)$, where $U_s(2)$ with $s = \uparrow, \downarrow$ is the sublattice-valley symmetry of quasiparticles with a fixed spin. In practice, however, the flavor $U(4)$ symmetry could be treated as exact because, for any realistic value of the magnetic field, the Zeeman energy is much smaller than the Landau energy scale. Moreover, the Zeeman energy, as well as all other explicit symmetry breaking effects are negligible not only compared to the Landau energy scale, but even compared to the nonzero (thermal/interaction) widths of the individual Landau levels. This explains why it is hard to resolve experimentally the QH plateaus with any filling factors other than $\nu = \pm 4(n + 1/2)$ that correspond to fully filled fourfold-degenerate Landau levels.

The subsequent experimental studies revealed that, in sufficiently strong magnetic fields, additional QH plateaus at $\nu = 0, \pm 1, \pm 3, \pm 4$ can be resolved [70, 85, 71, 123, 124,

125, 83, 84, 126, 127, 128, 129, 130]. As is clear, the corresponding QH states have fractional fillings of the low-lying Landau levels. The implication of this is that the approximate fourfold degeneracy of the Landau levels is truly lifted. Such new QH states could be explained theoretically if the electron-electron interaction triggers spontaneous breaking of the flavor $U(4)$ symmetry. For example, the symmetry breaking $U(4) \rightarrow U_{\uparrow}(2) \times U_{\downarrow}(2)$ by a *dynamically enhanced* Zeeman splitting could potentially explain the origin of the QH states with $\nu = 0$ and $\nu = \pm 4$. If realized, such a scenario would be nothing else, but a textbook example of the QH ferromagnetism (QHF) [74, 72, 77, 75, 131, 16] [see also Ref. [132]].

It should be emphasized, however, that QHF is not the only possibility. There exist a number of symmetry breaking mechanisms and residual symmetries consistent with the filling factors of the additional QH states. One of such alternative scenarios utilizes the idea of magnetic catalysis (MC) [21, 133, 134]. The corresponding order parameters are excitonic (particle-hole) condensates responsible for the generation of the Dirac and/or Haldane masses of quasiparticles [56, 135, 14, 87, 80, 12, 13, 86, 136]. At the microscopic level, the excitonic condensates corresponds to a charge density wave (CDW), or a valley polarized CDW. Symmetry arguments, as well as direct effective model studies [12, 13] suggest that the order parameters associated with the MC and QHF scenarios necessarily coexist. Also, in principle, these two could reproduce all integer QH plateaus observed experimentally in strong magnetic fields. With that being said, a number of different types of order parameters are possible [86, 136, 137, 138, 139, 140]. It is also fair to note that the precise nature of the observed integer QH states is not always unambiguous from theoretical considerations and not always established unambiguously in experiment.

From the experimental point of view, a lot of effort was devoted to revealing the underlying nature of the strongly insulating $\nu = 0$ QH state, associated with half-filling of the lowest Landau level [70, 85, 71, 123, 124, 125, 83, 84, 126, 127, 128, 129, 130]. The

main advances in resolving the nature of the QH states were made by applying a tilted magnetic field to high-quality graphene devices, that were fabricated on a thin hexagonal boron nitride substrate [128, 129] or encapsulated between two layers of hexagonal boron nitride [130]. In the case of $\nu = 0$ QH state, in particular, a careful analysis of the conductance and the bulk density of states [128, 129, 130] suggests that the state is not spin polarized when the magnetic field is perpendicular to the plane of graphene. This is consistent, for example, with both an antiferromagnet and a charge density wave. When an in-plane component of the magnetic field increases, the state gradually transforms into a fully polarized ferromagnetic state. In the intermediate regime, a canted antiferromagnetic state is presumably realized [137]. Such an interpretation is supported by the observation of the bulk gap that does not close and the edge states that become conducting with increasing of the total magnetic field. Considering, however, how heavily the arguments rely on the properties of the edge states, the final conclusions should be still accepted with a caution.

The high-quality graphene devices in Refs. [128, 130] also reveal a large sequence of integer QH states with $\nu \geq 1$. Among these, there are two states associated with quarter filling of the lowest Landau level, i.e., $\nu = \pm 1$. The linear growth of the energy gaps in these two states as a function of the total magnetic field points to their spin polarized nature and the major role played by the Zeeman energy in aligning the ground state. A similar sensitivity to the Zeeman energy is also observed for the states associated with half-filling of higher Landau levels, i.e., $\nu = 4n$, but not for the states with quarter and three-quarter fillings, i.e., $\nu = 4n \pm 1$ [128]. As we will see, most of these features are reproduced in the model studied here.

The role of the long-range Coulomb interaction and Landau-level mixing were studied in Ref. [89]. By utilizing a rather general combination of the MC and QHF order parameters, the corresponding study was able to reproduce all observed QH plateaus (i.e., $\nu = 0, \pm 1, \pm 3, \pm 4$) as well as to suggest that QH plateaus with any integer filling are possi-

ble. The symmetry breaking patterns of the solutions obtained in the model with the long-range Coulomb interaction appeared to be similar to those in the model with local four-fermion interaction in Ref. [13]. The qualitative effects due to the long-range force were (i) Landau-level mixing and (ii) “running” of all dynamical parameters (i.e., the wavefunction renormalization parameters, the Dirac and Haldane masses) as functions of the Landau-level index. In this chapter, we extend the study of the abnormal integer QH effect in the model with the long-range Coulomb interaction [89] by including the screening effects and the effects of nonzero temperature. In particular, by making use of a very flexible GLLR formalism, we derive the full quasiparticle propagators for all qualitatively different series of QH states with integer filling factors. As expected, the corresponding results contain not only the information about the energy gaps at the Fermi level, but also the complete dispersion relations of quasiparticles in all Landau levels. The resulting propagators can be used in transport calculations, predictions of various emission and absorption rates, etc.

This chapter is organized as follows. In Sec. 2.2, we briefly introduce the model, define the quasiparticle propagator, and derive the gap equations. In Sec. 2.3, we present our numerical analysis of the gap equations and classify the main types of solutions. We also compare our results with those in the previous studies. The summary of the main results is given in Sec. 2.4.

2.2 Model

Following the same approach as in Ref. [89], in this study we will use the language of the low-energy theory for the description of the QH effect in monolayer graphene. The low-energy quasiparticle fields are given by the following four-component Dirac spinors $\Psi_s = (\psi_{KAs}, \psi_{KBS}, \psi_{K'BS}, -\psi_{K'As})$, which combine the Bloch states on the two different sublattices (A, B) of the hexagonal graphene lattice in coordinate space. The components labeled by the valley indices K and K' correspond to the Bloch states with the momenta

from the vicinity of the corresponding inequivalent Dirac points (K or K') in the two-dimensional Brillouin zone. The field Ψ_s also carries an additional index that describes the spin state, i.e., $s = \uparrow, \downarrow$.

2.2.1 Quasiparticle Hamiltonian

The free quasiparticle Hamiltonian has the following pseudorelativistic form:

$$H_0 = v_F \int d^2\mathbf{r} \bar{\Psi} (\gamma^1 \pi_x + \gamma^2 \pi_y) \Psi, \quad (2.1)$$

where $v_F \approx 10^6 \text{m/s}$ is the Fermi velocity, $\mathbf{r} = (x, y)$ is the position vector in the plane of graphene, and $\bar{\Psi} = \Psi^\dagger \gamma^0$ is the Dirac-conjugated spinor. Note that the sum over the spin states is implicit in Eq. (2.1). The canonical momentum $\boldsymbol{\pi} = -i\hbar \nabla - e\mathbf{A}/c$ includes the vector potential $\mathbf{A}$ that corresponds to the magnetic field component $\mathbf{B}_\perp$ orthogonal to the plane of graphene. The 4×4 Dirac matrices γ^ν (with $\nu = 0, 1, 2$) are defined as follows: $\gamma^\nu = \tilde{\tau}^3 \otimes (\tau^3, i\tau^2, -i\tau^1)$, where the two sets of Pauli matrices $\tilde{\tau}$ and τ act in the valley (K, K') and sublattice (A, B) spaces, respectively. They satisfy the usual anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where $g^{\mu\nu} = \text{diag}(1, -1, -1)$, and $\mu, \nu = 0, 1, 2$.

In order to be able to describe QH states with arbitrary filling factors, we should also allow for a nonzero electron chemical potential μ_e . The latter is introduced via an additional term, $-\mu_e \int d^2\mathbf{r} \Psi^\dagger \Psi$, in the free Hamiltonian in Eq. (2.1). The long range Coulomb interaction is included in the low-energy Hamiltonian by adding the following term:

$$H_C = \frac{1}{2} \int d^2\mathbf{r} d^2\mathbf{r}' \Psi^\dagger(\mathbf{r}) \Psi(\mathbf{r}) U_C(\mathbf{r} - \mathbf{r}') \Psi^\dagger(\mathbf{r}') \Psi(\mathbf{r}'), \quad (2.2)$$

where $U_C(\mathbf{r} - \mathbf{r}')$ is the coordinate-space Coulomb potential in the presence of a constant magnetic field. It is straightforward to check that the resulting Hamiltonian $H = H_0 + H_C - \mu_e \int d^2\mathbf{r} \Psi^\dagger \Psi$ is invariant under the flavor $U(4)$ symmetry that combines the transformations in spin and sublattice-valley spaces. For the explicit form of the symmetry generators see, for example, Ref. [56]. Such a flavor symmetry is explicitly (although weakly) broken

by the Zeeman interaction of the quasiparticle spin with the external magnetic field $\mathbf{B}$. The additional interaction term in the Hamiltonian is given by $\int d^2\mathbf{r} \mu_B B \Psi^\dagger \sigma^3 \Psi$, where $B = |\mathbf{B}|$, $\mu_B = e\hbar/(2mc)$ is the Bohr magneton, and σ^3 is the Pauli matrix acting in the spin space. As is easy to check, the Zeeman term breaks the $U(4)$ symmetry down to the $U_\uparrow(2) \times U_\downarrow(2)$ symmetry.

The complete model Hamiltonian, including the Zeeman interaction and the electron chemical potential μ_e , can be conveniently rewritten in the following quasirelativistic form:

$$H = \int d^2r \bar{\Psi} \left[v_F (\boldsymbol{\pi} \cdot \boldsymbol{\gamma}) - \mu_e \gamma^0 + \mu_B B \sigma^3 \gamma^0 \right] \Psi + H_C. \quad (2.3)$$

This implies that the inverse bare quasiparticle propagator is defined by the following expression:

$$iS^{-1}(\omega; \mathbf{r}, \mathbf{r}') = \left[(\omega + \mu_e - \mu_B B \sigma^3) \gamma^0 - v_F (\boldsymbol{\pi} \cdot \boldsymbol{\gamma}) \right] \delta(\mathbf{r} - \mathbf{r}'). \quad (2.4)$$

Here and below, it is convenient to use a mixed $(\omega, \mathbf{r})$ -representation.

Let us note in passing that a certain degree of disorder is always present in real graphene devices and, in fact, plays a critical role in the observation of the quantum Hall plateaus. However, in the model analysis below, which concentrates primarily on the delocalized quasiparticles states in the bulk of graphene, it is justifiable to ignore disorder. Indeed, when the spectrum of the delocalized states is established, a large number of bulk observables (e.g., symmetry properties of the QH states, density of states, energy of transitions lines, activation energies, etc.) will be predicted without much ambiguity. The corresponding limitation is not so critical also because of the robustness of the QH effect that stem from its topological nature.

2.2.2 Full Quasiparticle Propagator

In order to be able to describe various QH states with integer fillings factors, we use a rather general ansatz for the full quasiparticle propagator [89],

$$G^{-1}(\omega; \mathbf{r}, \mathbf{r}') = -i \left[(\gamma^0 \omega - v_F \hat{F}^+ (\boldsymbol{\pi} \cdot \boldsymbol{\gamma}) + \hat{\Sigma}^+) \right] \delta(\mathbf{r} - \mathbf{r}'), \quad (2.5)$$

where $\hat{F}^+$ and $\hat{\Sigma}^+$ are operator-valued wavefunction renormalization and self-energy functions, respectively. For simplicity, here we will assume that the propagator is a diagonal 2×2 matrix in the spinor space, or in other words that the propagator for each of the two spin states looks like that in Eq. (2.5). As is clear, such a choice of the ansatz for the propagator does not allow spin mixing. As a result, one cannot describe certain types of states (e.g., a canted antiferromagnetism [137]). Nevertheless, we emphasize that the ansatz in Eq. (2.5) is extremely flexible and can describe a large number of states (with very different symmetry properties) for each integer filling factor. In the case of filling factor $\nu = 0$, for example, these include a charge/spin density waves, as well as ferromagnetic and antiferromagnetic states. The latter are basically all main options proposed for a configuration with a magnetic field perpendicular to the plane of graphene.

By construction, both $\hat{F}^+$ and $\hat{\Sigma}^+$ are functions of the three mutually commuting dimensionless operators: γ^0 , $is_{\perp} \gamma^1 \gamma^2$, and $(\boldsymbol{\pi} \cdot \boldsymbol{\gamma})^2 l^2$, where $s_{\perp} = \text{sign}(eB)$ and $l = \sqrt{\hbar c / |eB_{\perp}|}$ is the magnetic length. In principle, they could also depend on the quasiparticle energy ω . Taking into account that $(\gamma^0)^2 = \mathbb{I}$ and $(is_{\perp} \gamma^1 \gamma^2)^2 = \mathbb{I}$, the Dirac structure of the operator-valued functions $\hat{F}^+$ and $\hat{\Sigma}^+$ can be written in the following form:

$$\hat{F}^+ = f + \gamma^0 g + is_{\perp} \gamma^1 \gamma^2 \tilde{g} + is_{\perp} \gamma^0 \gamma^1 \gamma^2 \tilde{f}, \quad (2.6)$$

$$\hat{\Sigma}^+ = \tilde{\Delta} + \gamma^0 \mu + is_{\perp} \gamma^1 \gamma^2 \tilde{\mu} + is_{\perp} \gamma^0 \gamma^1 \gamma^2 \Delta, \quad (2.7)$$

where f , $\tilde{f}$, g , $\tilde{g}$, $\tilde{\Delta}$, Δ , μ , and $\tilde{\mu}$ are functions of only one operator, $(\boldsymbol{\pi} \cdot \boldsymbol{\gamma})^2 l^2$. To a large degree the physical meaning of the corresponding operators should be clear from their

Dirac structure and symmetry properties. In particular, the eigenvalues of the first four of them ($f, \tilde{f}, g$, and $\tilde{g}$) will play the role of generalized wavefunction renormalizations, while the others will be playing the role of the generalized the MC and QHF order parameters, i.e., the Dirac (parity-even) and Haldane (parity-odd) masses (i.e., Δ and $\tilde{\Delta}$) and chemical potentials (i.e., μ and $\tilde{\mu}$). As will see, such an interpretation is also supported by the role of the corresponding parameters in the dispersion relations of quasiparticles, see Eqs. (2.13) and (2.14) below.

The three mutually commuting operators, γ^0 , $is_\perp \gamma^1 \gamma^2$, and $(\boldsymbol{\pi} \cdot \boldsymbol{\gamma})^2 l^2$, allow a common basis of eigenstates, $|s_0, s_{12}, n\rangle$. The corresponding eigenvalues are $s_0 = \pm 1$, $s_{12} = \pm 1$, and $-2n = -(2N + 1 + s_\perp s_{12})$, respectively, where $N = 0, 1, 2, \dots$ is the quantum number of the orbital angular momentum and s_{12} is the sign of the pseudospin projection. By making use of the complete set of eigenstates $|s_0, s_{12}, n\rangle$, one can derive a very convenient generalized Landau-level representation (GLLR) of the (inverse) quasiparticle propagator (2.5). (For details of the derivation, see Appendix A in Ref. [89].) The final form of the *inverse* full propagator is given by

$$G^{-1}(\omega; \mathbf{r}, \mathbf{r}') = e^{i\Phi(\mathbf{r}, \mathbf{r}')} \tilde{G}^{-1}(\omega; \mathbf{r} - \mathbf{r}'), \quad (2.8)$$

where $\Phi(\mathbf{r}, \mathbf{r}') = -s_\perp \frac{(x+x')(y-y')}{2l^2}$ is the well-known Schwinger phase in an external magnetic field in the Landau gauge $\mathbf{A} = (0, Bx)$. The translation invariant part of the inverse GLLR propagator is given by

$$\begin{aligned} \tilde{G}^{-1}(\omega; \mathbf{r}) = & -i \frac{e^{-\xi/2}}{2\pi l^2} \sum_{n=0}^{\infty} \sum_{s_0=\pm 1} \sum_{\sigma=\pm 1} \left\{ s_0 \omega L_n(\xi) \right. \\ & + (s_0 \mu_{n,\sigma} + \tilde{\Delta}_{n,\sigma}) [\delta_{-\sigma}^{s_0} L_n(\xi) + \delta_{+\sigma}^{s_0} L_{n-1}(\xi)] \\ & \left. + i \frac{v_F^2}{l^2} (\boldsymbol{\gamma} \cdot \mathbf{r}) (f_{n,\sigma} - s_0 g_{n,\sigma}) L_{n-1}^1(\xi) \right\} \mathcal{P}_{s_0, s_0 \sigma}, \end{aligned} \quad (2.9)$$

where $\xi = \mathbf{r}^2 / (2l^2)$ and L_n^α are the Laguerre polynomials. (Here, by definition, $L_n^0 \equiv L_n$ and $L_{n<0}^\alpha = 0$.) In Eq. (2.9), we also used the following set of eigenstate projectors in the Dirac

space: $P_{s_0, s_{12}} = \frac{1}{4}(1 + s_0 \gamma^0)(1 + s_{12} i s_\perp \gamma^1 \gamma^2)$, as well as the following shorthand notations for the linear combinations of the order parameters:

$$\begin{aligned}\mu_{n, \sigma} &= \mu_n + \sigma \tilde{\mu}_n, & \tilde{\Delta}_{n, \sigma} &= \tilde{\Delta}_n + \sigma \Delta_n, \\ f_{n, \sigma} &= f_n + \sigma \tilde{f}_n, & g_{n, \sigma} &= g_n + \sigma \tilde{g}_n,\end{aligned}\tag{2.10}$$

where $\sigma \equiv s_0 s_{12}$. Note that the parameters f_n , $\tilde{f}_n$, g_n , $\tilde{g}_n$, $\tilde{\Delta}_n$, Δ_n , μ_n , and $\tilde{\mu}_n$ are associated with the n th Landau level. They are obtained by calculating the eigenvalues of the corresponding operators, introduced on the right-hand side of Eqs. (2.6) and (2.7).

Similarly, the full GLLR propagator takes the form [89]

$$G(\omega; \mathbf{r}, \mathbf{r}') = e^{i\Phi(\mathbf{r}, \mathbf{r}')} \tilde{G}(\omega; \mathbf{r} - \mathbf{r}'),\tag{2.11}$$

where the Schwinger phase is exactly the same as in Eq. (2.8) and the translation invariant part of the propagator reads

$$\begin{aligned}\tilde{G}(\omega; \mathbf{r}) &= i \frac{e^{-\xi/2}}{2\pi l^2} \sum_{n=0}^{\infty} \sum_{s_0=\pm 1} \sum_{\sigma=\pm 1} \left\{ \frac{s_0(\omega + \mu_{n, \sigma}) + \tilde{\Delta}_{n, \sigma}}{(\omega + \mu_{n, \sigma})^2 - E_{n, \sigma}^2} \right. \\ &\times [\delta_{-\sigma}^{s_0} L_n(\xi) + \delta_{+\sigma}^{s_0} L_{n-1}(\xi)] \\ &+ \left. i \frac{v_F^2}{l^2} (\boldsymbol{\gamma} \cdot \mathbf{r}) \frac{f_{n, \sigma} - s_0 g_{n, \sigma}}{(\omega + \mu_{n, \sigma})^2 - E_{n, \sigma}^2} L_{n-1}^1(\xi) \right\} \mathcal{P}_{s_0, s_0 \sigma},\end{aligned}\tag{2.12}$$

The explicit form of the Landau level energies $E_{n, \sigma}$ are given by

$$E_{0, \sigma} = \sigma \tilde{\Delta}_{0, \sigma} = \Delta_0 + \sigma \tilde{\Delta}_0,\tag{2.13}$$

$$E_{n, \sigma} = \sqrt{2n \left(f_{n, \sigma}^2 - g_{n, \sigma}^2 \right) v_F^2 / l^2 + \tilde{\Delta}_{n, \sigma}^2}, \quad n \geq 1.\tag{2.14}$$

The corresponding quasiparticles energies are determined by the location of the poles of the propagator (2.12), namely $\omega_{0, \sigma} = -\mu_{0, \sigma} + E_{0, \sigma}$ and $\omega_{n, \sigma}^\pm = -\mu_{n, \sigma} \pm E_{n, \sigma}$ for $n \geq 1$.

Note that the expressions for the Landau level energies in Eqs. (2.13) and (2.14) shed additional light on the physical meaning of the various dynamical parameters, used in

the ansatz of the full quasiparticle propagator. In particular, as we can see, the combination of the wavefunction renormalization parameters $\sqrt{f_{n,\sigma}^2 - g_{n,\sigma}^2}$ determines the renormalization of the quasiparticle velocity parameter. Also, we see that the absolute value of $\tilde{\Delta}_{n,\sigma}$ plays the role of a mass.

2.2.3 *Schwinger-Dyson Equation with Static Screening*

In order to derive the GLLR form of the Schwinger-Dyson (gap) equation, we will start from the standard coordinate form of the gap equation in the random-phase approximation (RPA) and assume that the interaction is provided by the long-range Coulomb force subject to static screening effects. Such a consideration will amend the analysis of Ref. [89], where the corresponding gap equation was analyzed in the approximation without screening. Our goal here is to illuminate the qualitative and quantitative role played by screening.

It is important to emphasize that we will use a mean-field method that ignores the fluctuations of order parameters. While this is a common approximation used in numerous theoretical studies, there is no rigorous justification of its validity. Formally, the corresponding fluctuations can destroy any long-range order in 2D when $T \neq 0$ and, thus, prevent any spontaneous breakdown of continuous symmetries. We argue semi-rigorously that this may not be the case for finite-size, high-quality graphene devices fabricated on substrates. The substrate could make certain aspects of dynamics in graphene effectively three-dimensional and, thus, tame the dangerous fluctuations (at least on the length scales of typical devices) that would destroy the long-range order in an ideal 2D graphene. Admittedly, however, this issue requires a more careful investigation in the future studies.

After taking into account the condition of overall charge neutrality in graphene, we arrive at the following gap equation for the translation invariant part of the quasiparticle propagator [89]:

$$\tilde{G}^{-1}(\omega; \mathbf{r}) = \tilde{S}^{-1}(\omega; \mathbf{r}) + e^2 \gamma^0 \tilde{G}(\omega; \mathbf{r}) \gamma^0 D(\omega; \mathbf{r}), \quad (2.15)$$

where $D(\omega; \mathbf{r})$ is the time-like component of the gauge (photon) field propagator that contains all the information about the interaction. As stated before, we will use an approximation with Coulomb interaction that includes the effects of static screening. In essence, this is the instantaneous approximation [56] that neglects the retardation of the interaction. One might argue that this is a reasonable approximation because charge carriers are much slower than the speed of light. (See, however, Refs. [141, 142] suggesting that dynamical screening could affect the dynamics quantitatively.) Therefore, we use an energy independent polarization function, i.e., $\Pi(\omega, \mathbf{k}) \simeq \Pi(0, \mathbf{k})$, in order to model the effects of screening in the photon propagator. In momentum space, the latter reads

$$D(\omega, \mathbf{k}) \approx D(0, \mathbf{k}) = \frac{i}{\epsilon_0 [k + \Pi(0, \mathbf{k})]}, \quad (2.16)$$

where ϵ_0 is the dielectric constant, associated with the substrate. When the value of ϵ_0 is large, the underlying dynamics becomes weakly coupled and the approximation for the gap equation (2.15) should become reliable. This is not always the case in graphene. However, we expect that this may be applicable in the case of the highest quality graphene devices, fabricated on a thin hexagonal boron nitride substrate [128].

A convenient explicit form of the polarization function in the presence of an external magnetic field was calculated in Refs. [141]. In the approximation that neglects the wavefunction renormalization and the Dirac masses, it reads

$$\begin{aligned} \Pi(0, \mathbf{k}) = & \frac{e^2 N_f}{16\pi l^2 T} \sum_{n=0}^{\infty} \sum_{\lambda=\pm} \frac{Q_{nn}^{\lambda\lambda}(y)}{\cosh^2(\frac{\mu - \lambda M_n}{2T})} - \frac{e^2 N_f}{4\pi l^2} \sum_{\substack{n, n'=0 \\ \lambda n \neq \lambda' n'}}^{\infty} \sum_{\lambda, \lambda'=\pm} \\ & Q_{nn'}^{\lambda\lambda'}(y) \frac{n_F(\lambda M_n) - n_F(\lambda' M_{n'})}{\lambda M_n - \lambda' M_{n'}} \end{aligned} \quad (2.17)$$

where $M_n = \sqrt{2m_F^2/l^2}$, T is the temperature, and the explicit form of function $Q_{n,n'}^{\lambda,\lambda'}(y)$ is

given by [141]

$$\begin{aligned}
Q_{n,n'}^{\lambda,\lambda'}(y) &= e^{-y} y^{|n-n'|} \left(\sqrt{\frac{(1+\lambda\lambda'\delta_{0n_{>}})n_{<}!}{n_{>}!}} L_{n_{<}}^{|n-n'|}(y) \right. \\
&\quad \left. + \lambda\lambda'(1-\delta_{0n_{<}}) \sqrt{\frac{(n_{<}-1)!}{(n_{>}-1)!}} L_{n_{<}-1}^{|n-n'|}(y) \right)^2.
\end{aligned} \tag{2.18}$$

Here, by definition, $y = \mathbf{k}^2 l^2 / 2$, $n_{>} = \max(n, n')$ and $n_{<} = \min(n, n')$. We note that neglecting the wavefunction renormalization and masses in the calculation of the polarization function should provide a reasonable approximation to leading order in weak coupling. Moreover, this may work also at moderate coupling because the bulk of the polarization effects appear to be determined by the total number of filled Landau levels and not as much by the details of the quasiparticle dispersion relations.

For the derivation of the GLLR form of the gap equations, we refer the reader to Appendix B in Ref. [89]. The final set of equations reads

$$\begin{aligned}
\mu_{n,\sigma} - \mu_e - \sigma \tilde{\Delta}_{n,\sigma} &= \frac{\alpha \varepsilon_l}{2} \sum_{n'=0}^{\infty} \kappa_{n',n}^{(0)} \left\{ n_F(E_{n',\sigma} - \mu_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma}) \right. \\
&\quad \left. - \frac{\sigma \tilde{\Delta}_{n',\sigma}}{E_{n',\sigma}} [n_F(\mu_{n',\sigma} - E_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma})] \right\},
\end{aligned} \tag{2.19}$$

for $n \geq 0$, and

$$\begin{aligned}
\mu_{n,\sigma} - \mu_e + \sigma \tilde{\Delta}_{n,\sigma} &= \frac{\alpha \varepsilon_l}{2} \sum_{n'=1}^{\infty} \kappa_{n'-1,n-1}^{(0)} \left\{ n_F(E_{n',\sigma} - \mu_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma}) \right. \\
&\quad \left. + \frac{\sigma \tilde{\Delta}_{n',\sigma}}{E_{n',\sigma}} [n_F(\mu_{n',\sigma} - E_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma})] \right\},
\end{aligned} \tag{2.20}$$

$$f_{n,\sigma} = 1 + \frac{\alpha \varepsilon_l}{2} \sum_{n'=1}^{\infty} \frac{\kappa_{n'-1,n-1}^{(1)}}{n} \frac{f_{n',\sigma}}{E_{n',\sigma}} [n_F(\mu_{n',\sigma} - E_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma})], \tag{2.21}$$

$$g_{n,\sigma} = \frac{\alpha \varepsilon_l}{2} \sum_{n'=1}^{\infty} \frac{\kappa_{n'-1,n-1}^{(1)}}{n} \frac{g_{n',\sigma}}{E_{n',\sigma}} [n_F(\mu_{n',\sigma} - E_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma})], \tag{2.22}$$

for $n \geq 1$. In these equations, we introduced a dimensionless coupling constant, $\alpha \equiv e^2 / \varepsilon_0 v_F \approx 2.2$, which is an analog of the fine structure constant for suspended graphene.

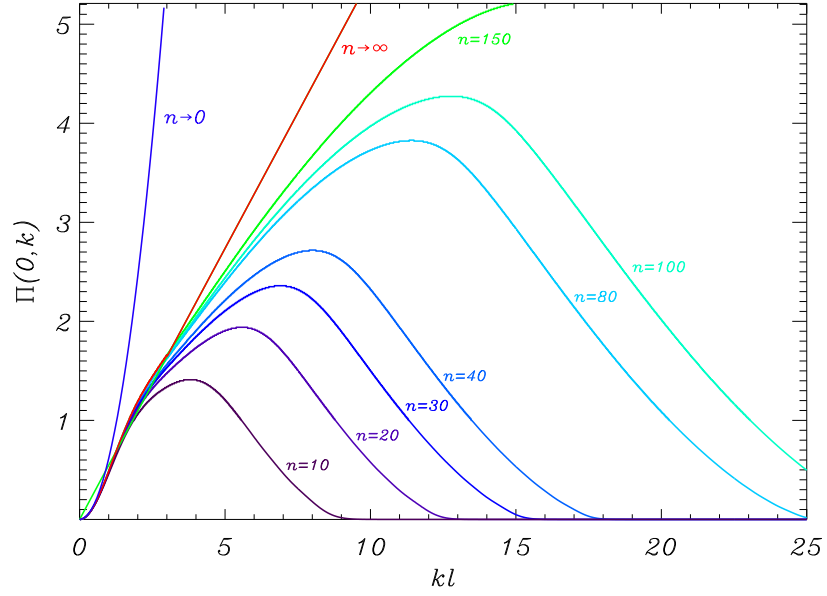


Figure 2.1: The static polarization function $\Pi(0, k)$ as a function of dimensionless variable kl with the increase of the maximum Landau level n in the summation.

The effect of screening in the above set of gap equations is implicit. It comes only through the modified values of the kernel coefficients [89],

$$\kappa_{m,n}^{(\rho)} = \int_0^\infty \frac{dk}{2\pi} \frac{kl \mathcal{L}_{m,n}^{(\rho)}(kl)}{k + \Pi(0, k)}, \quad \text{with } \rho = 0, 1. \quad (2.23)$$

It is straightforward to calculate numerically the static polarization function $\Pi(0, k)$ as a function of dimensionless variable kl . We checked that, for sufficiently small and sufficiently large values of kl , the numerical results (Fig. 2.1) approach the expected asymptotical behavior at $n \rightarrow 0$ and $n \rightarrow \infty$, respectively, derived in Eqs. (19) and (22) in Ref. [141]. Then, by making use of the definition in Eq. (2.23), we obtain the numerical values of the kernel coefficients $\kappa_{m,n}^{(\rho)}$ (with $\rho = 0, 1$). The results for the first few Landau levels are presented in Tables 2.1 and 2.2, and Fig. 2.2 respectively. We see that the large values of the kernel coefficients distribute along the diagonal, the values decrease with the increase of the order (m, n) for $\kappa_{m,n}^{(0)}$ and increase on the opposite for $\kappa_{m,n}^{(1)}$. By comparing these with

Table 2.1: The kernel coefficients $\kappa_{m,n}^{(0)}$ with the effects of static screening included.

$\kappa_{m,n}^{(0)}$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
0	0.0744	0.0235	0.0164	0.0136	0.0120	0.0109	0.0101	0.0094	0.0088	0.0083	0.0079
1	0.0235	0.0602	0.0222	0.0156	0.0129	0.0114	0.0104	0.0096	0.0090	0.0085	0.0081
2	0.0164	0.0222	0.0531	0.0211	0.0149	0.0124	0.0109	0.0100	0.0093	0.0087	0.0083
3	0.0136	0.0156	0.0211	0.0486	0.0202	0.0144	0.0120	0.0106	0.0097	0.0090	0.0085
4	0.0120	0.0129	0.0149	0.0202	0.0453	0.0195	0.0140	0.0116	0.0103	0.0094	0.0088
5	0.0109	0.0114	0.0124	0.0144	0.0195	0.0428	0.0189	0.0137	0.0113	0.0100	0.0092
6	0.0101	0.0104	0.0109	0.0120	0.0140	0.0189	0.0407	0.0184	0.0133	0.0111	0.0098
7	0.0094	0.0096	0.0100	0.0106	0.0116	0.0137	0.0184	0.0390	0.0180	0.0131	0.0109
8	0.0088	0.0090	0.0093	0.0097	0.0103	0.0113	0.0133	0.0180	0.0375	0.0176	0.0128
9	0.0083	0.0085	0.0087	0.0090	0.0094	0.0100	0.0111	0.0131	0.0176	0.0362	0.0172
10	0.0079	0.0081	0.0083	0.0085	0.0088	0.0092	0.0098	0.0109	0.0128	0.0172	0.0351

the kernel coefficients in Ref. [89], we observe that the screening effects substantially (i.e., by about a factor of 2) decrease the values of $\kappa_{m,n}^{(\rho)}$. As we will see in the next section, this change strongly affects the magnitude of the order parameters (i.e., the symmetry breaking Dirac masses and chemical potentials) in the QH states with fractional filling of Landau levels.

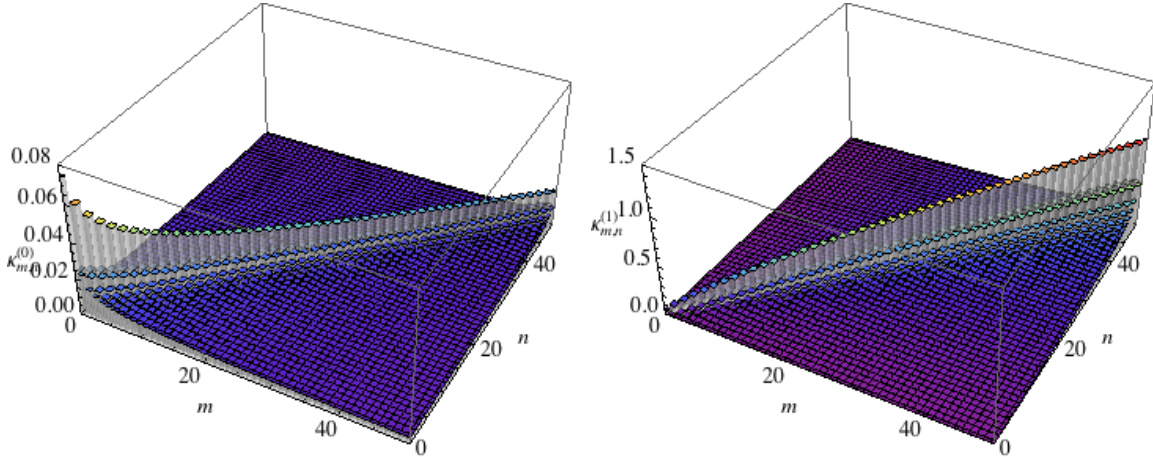


Figure 2.2: Left panel: the kernel coefficients $\kappa_{m,n}^{(0)}$ with the effects of static screening included. Right panel: the kernel coefficients $\kappa_{m,n}^{(1)}$ with the effects of static screening included.

Table 2.2: The kernel coefficients $\kappa_{m,n}^{(1)}$ with the effects of static screening included.

$\kappa_{m,n}^{(1)}$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
0	0.0509	0.0142	0.0084	0.0064	0.0055	0.0051	0.0048	0.0046	0.0043	0.0041	0.0039
1	0.0142	0.0901	0.0283	0.0172	0.0131	0.0111	0.0099	0.0093	0.0088	0.0084	0.0081
2	0.0084	0.0283	0.1244	0.0419	0.0260	0.0198	0.0167	0.0148	0.0137	0.0129	0.0123
3	0.0064	0.0172	0.0419	0.1553	0.0550	0.0346	0.0264	0.0222	0.0197	0.0181	0.0169
4	0.0055	0.0131	0.0260	0.0550	0.1839	0.0677	0.0431	0.0330	0.0276	0.0244	0.0224
5	0.0051	0.0111	0.0198	0.0346	0.0677	0.2107	0.0801	0.0515	0.0395	0.0331	0.0292
6	0.0048	0.0099	0.0167	0.0264	0.0431	0.0801	0.2361	0.0922	0.0597	0.0459	0.0385
7	0.0046	0.0093	0.0148	0.0222	0.0330	0.0515	0.0922	0.2602	0.1039	0.0678	0.0523
8	0.0043	0.0088	0.0137	0.0197	0.0276	0.0395	0.0597	0.1039	0.2834	0.1154	0.0758
9	0.0041	0.0084	0.0129	0.0181	0.0244	0.0331	0.0459	0.0678	0.1154	0.3056	0.1266
10	0.0039	0.0081	0.0123	0.0169	0.0224	0.0292	0.0385	0.0523	0.0758	0.1266	0.3271

2.3 Numerical Results

In this section, we present our numerical solutions to the GLLR gap equations (2.19) through (2.22) in the model with static screening effects. In the calculation, we use the Newtonian iteration algorithm and replace an original infinite set of gap equations with a truncated set of $n_{\max} = 50$ equations. In accordance with such a truncation, we also impose a sharp cutoff in the summation over the Landau level index at $n_{\max} = 50$. (The numerical tests reveal that the cutoff at $n_{\max} = 100$ does not affect any qualitative features of the solutions and only slightly changes the numerical results for the dynamical parameters in the first few Landau levels.)

Taking into account that each Landau level ($n \leq n_{\max}$) has 2 possible spin states and each of them is characterized by 8 different dynamical parameters, see Eq. (2.10), we have a total of $16n_{\max}$ independent parameters that should be determined by solving $16n_{\max}$ coupled algebraic equations, see Eq. (2.19) through (2.22). [Strictly speaking the number of independent parameters is $16n_{\max} + 4$, where the additional 4 corresponds to the $n = 0$ Landau level, which is special.]

Before proceeding with the numerical analysis, it is useful to note that the coupled sets of gap equations for the two spin states, $s = \downarrow, \uparrow$, have exactly the same form. They

differ only in the value of the chemical potential, i.e., $\mu_\uparrow = \mu_e - Z$ and $\mu_\downarrow = \mu_e + Z$, where $Z \equiv \mu_B B$ is the Zeeman energy. We also note that the gap equations are the same for $\sigma = \pm 1$ when the dynamical parameters $f_{n,\sigma}$, $g_{n,\sigma}$, $\mu_{n,\sigma}$ and $\sigma \tilde{\Delta}_{n,\sigma}$ are treated as independent variables. These two observations allow us to greatly reduce the numerical cost of calculations. We first consider the generic set of equations in a finite range of chemical potentials and tabulate possible solutions for the above-mentioned four independent parameters. Usually, there exist multiple solutions that differ in their symmetry properties. The final solutions for the complete set of dynamical parameters with fixed values of $s = \downarrow, \uparrow$ and $\sigma = \pm 1$ are obtained by combining the distinct generic solutions obtained at properly shifted values of the chemical potential.

In the numerical analysis, it is convenient to express all physical quantities in units of the Landau energy scale,

$$\varepsilon_l = \sqrt{\hbar v_F^2 |eB_\perp|/c} \approx 26 \sqrt{B_\perp [\text{T}]} \text{ meV} \approx 300 \sqrt{B_\perp [\text{T}]} \text{ K}, \quad (2.24)$$

where the value of the magnetic field is measured in Teslas. It may be appropriate to emphasize here that, while ε_l is determined by the component of the magnetic field orthogonal to the plane of graphene $\mathbf{B}_\perp$, the Zeeman energy $Z = \mu_B B \approx 5.8 \times 10^{-2} B [\text{T}] \text{ meV} \approx 0.67 B [\text{T}] \text{ K}$ is proportional to the absolute value of $\mathbf{B}$. Despite the fact that $B_\perp \leq B$, the Zeeman energy is generically much smaller than the Landau energy scale (2.24). This changes only in the case when the magnetic field becomes nearly parallel to the plane of graphene, i.e., when $B_\perp \ll B$. In the analysis below, we do not consider such a limiting case. In general, the dimensionless Zeeman energy is given by $z \equiv Z/\varepsilon_l = 2.2 \times 10^{-3} \sqrt{B [\text{T}]/\cos \theta_B}$, where θ_B is the angle of the magnetic field tilt. In the numerical analysis below, we will fix the value of the dimensionless Zeeman energy to be $z = 0.015$. This formally corresponds to $B/\cos \theta_B \approx 46.5 \text{ T}$.

2.3.1 Fermi Velocity Renormalization in Weak Magnetic Field

In a weak magnetic field, the effect of symmetry breaking dynamical parameters is expected to be negligible. However, even in this regime it is interesting to explore the implications of the long-range Coulomb interaction on the quantum Hall effect in graphene. In particular, the interaction is expected to renormalize the Fermi velocity, which can be extracted from the dynamically modified expressions for the Landau level energies [89]. In absence of QHF and MC order parameters, as we see from Eq. (2.14), the energies are given by $E_n = f_n v_F \sqrt{2n} \varepsilon_l$, implying that the renormalized Fermi velocity is determined by the wavefunction renormalization parameters f_n , namely $v_{n,F}^{(ren)} \equiv f_n v_F$. Without screening of the Coulomb interaction, the numerical values for the wavefunction renormalization parameters f_n were previously reported in Ref. [89]. The corresponding results with the effects of screening were reported in Ref. [134] for the QH states with filling factors $\nu = \pm 4(k + 1/2)$. In the latter study, in fact, the dynamical parameters μ_n and Δ_n were also properly accounted for. By noting that the values of f_n depend not only on the chemical potential, but also on the Landau level index n , we conclude that the same is also true for the renormalized Fermi velocity. This is a very interesting theoretical prediction that could be easily tested in optical experiments, for example, via a systematic study of absorption/transmission lines for each of the QH states [143, 144, 145].

In the context of the optical transitions, the effect of interactions can be conveniently quantified by measuring the deviations of the measured energies of transitions from the free theory predictions [144],

$$\Delta E_{n,n'} = E_{n'} \pm E_n = (\sqrt{2n'} \pm \sqrt{2n}) \varepsilon_l + \alpha \varepsilon_l C_{n,n'}, \quad (2.25)$$

where the minus sign corresponds to transitions between states with negative energies and states with positive energies. Strictly speaking, the above definition of $\Delta E_{n,n'}$ assumes transitions between states with the same spin. In the case of transitions between different

Table 2.3: Values of the wavefunction renormalization f_n for several values of the chemical potentials with screening effects considered, compared with the solutions in [89] with screening effects neglected in the parentheses.

	f_1	f_2	f_3	f_4	f_5	f_6
$ \mu_c < \sqrt{2}\varepsilon_l$	1.084 (1.270)	1.072 (1.243)	1.065 (1.227)	1.060 (1.214)	1.057 (1.205)	1.054 (1.197)
$\sqrt{2}\varepsilon_l < \mu_c < \sqrt{4}\varepsilon_l$	1.045 (1.194)	1.066 (1.224)	1.063 (1.217)	1.059 (1.208)	1.056 (1.201)	1.054 (1.194)
$\sqrt{4}\varepsilon_l < \mu_c < \sqrt{6}\varepsilon_l$	1.037 (1.193)	1.042 (1.224)	1.058 (1.217)	1.057 (1.208)	1.055 (1.201)	1.052 (1.194)
$\sqrt{6}\varepsilon_l < \mu_c < \sqrt{8}\varepsilon_l$	1.033 (1.166)	1.035 (1.177)	1.039 (1.200)	1.052 (1.199)	1.052 (1.194)	1.051 (1.189)
$\sqrt{8}\varepsilon_l < \mu_c < \sqrt{10}\varepsilon_l$	1.031 (1.150)	1.032 (1.156)	1.034 (1.165)	1.037 (1.184)	1.048 (1.185)	1.049 (1.182)
$\sqrt{10}\varepsilon_l < \mu_c < \sqrt{12}\varepsilon_l$	1.029 (1.149)	1.030 (1.155)	1.031 (1.164)	1.032 (1.182)	1.035 (1.184)	1.045 (1.182)
$\sqrt{12}\varepsilon_l < \mu_c < \sqrt{14}\varepsilon_l$	1.027 (1.138)	1.028 (1.142)	1.028 (1.148)	1.029 (1.156)	1.031 (1.172)	1.034 (1.174)

spin states, an extra $\pm 2Z$ correction should be added on the right hand side. Here, the information about the wavefunction renormalization is captured by the following set of dimensionless parameters $C_{n,n'}$, i.e.,

$$C_{n,n'} = \frac{\sqrt{2n'}}{\alpha}(f_{n'} - 1) \pm \frac{\sqrt{2n}}{\alpha}(f_n - 1). \quad (2.26)$$

As we see, in absence of symmetry breaking, which is the case in weak magnetic fields, the values of parameters $C_{n,n'}$ are directly related to the quasiparticle velocity renormalizations. As we claim here, for each QH state, the corresponding renormalizations are functions of the Landau level index n . In experiment, the complete set of parameters f_n with $n \geq 1$ could be extracted by measuring the transition energies between the lowest Landau level (which is free from the corresponding renormalization effect) and higher Landau levels. The values of f_n , extracted in this way, would be sufficient to calculate the values of $C_{n,n'}$ for transitions between various higher Landau levels. The latter could be also compared to the actual measurements and, in the case of agreement, one would have a nontrivial test of the self-consistency of the GLLR ansatz used here. From a theoretical point of view, it will be perhaps even more interesting if deviations from the relations in Eq. (2.26) are observed.

In a general case with nonvanishing QHF and MC order parameters, the expressions for $C_{n,n'}$ parameters should be corrected because the Landau level energies are modified, see Eqs. (2.13) and (2.14). The magnitude of the corresponding corrections is of the order

of $|\mu_{n,\sigma} - \mu_e|$ and $\tilde{\Delta}_{n,\sigma}^2/(\sqrt{n}\epsilon_l)$ for Landau levels $n \geq 1$. (Note that the correction due to the QHF order parameter should be interpreted as part of the energy measured with respect to the thermodynamical potential μ_e .) In the case of transitions to/from the lowest Landau level ($n = 0$), the corrections are $\pm\sigma\tilde{\Delta}_{0,\sigma}$, see Eq. (2.13). Below, we present our numerical results for the parameters $C_{n,n'}$ in several QH states in both approximations, i.e., with and without inclusion of the QHF and MC order parameters.

By utilizing the same value of the cutoff, $n_{\max} = 100$, in the numerical calculation as in [89], but also including the effects of screening, we straightforwardly obtain the wavefunction renormalization parameters f_n . The corresponding zero temperature results for the first few Landau levels are presented in Table 2.3. We note that they differ slightly from those in Ref. [134] because here all dynamical parameters such as μ_n and Δ_n were neglected. For comparison, in Table 2.3 we also list in the parentheses the numerical results from Ref. [89], obtained without the screening effects. Different rows in Table 2.3 correspond to different choices of the chemical potential in the gaps between a completely filled n th Landau level and a completely empty $(n+1)$ th Landau level.

By comparing the results with and without screening in Table 2.3, we find that the effect of wave function renormalization goes from about 14% to 27% down to about 3% to 8%, which appears to be even smaller than the prediction in Ref. [146]. It is curious to explore in detail if the logarithmic running of the coupling constant could explain such a difference. By making use of the definition in Eq. (2.26) and our numerical results for various QH states, we readily calculate $C_{n,n'}$ parameters.

The representative results for $C_{n,n'}$ parameters are presented in Tables 2.4 and 2.5 for the QH states with the filling factors $\nu = 2$ and $\nu = 6$. Because of the particle-hole symmetry, the results for the corresponding negative filling factors can be obtained as follows: $C_{n,n'}(-\nu) = C_{-n',-n}(\nu)$. These results appear to be about 2 to 4 times smaller than the results in the absence of screening [89]. In calculation, we took into account the effect

Table 2.4: Parameters $C_{n,n'}$ in the QH state with the filling factor $\nu = 2$ in the model with static screening. The values in parentheses are obtained in an approximation with the wavefunction renormalization effects included, but all QHF and MC order parameters neglected.

$C_{n,n'}$	$n' = 1$	$n' = 2$	$n' = 3$	$n' = 4$
$n = 0$	0.082 (0.054)	0.093 (0.065)	0.099 (0.072)	0.103 (0.077)
$n = -1$	0.100 (0.108)	0.111 (0.119)	0.117 (0.126)	0.122 (0.131)
$n = -2$	0.108 (0.119)	0.119 (0.130)	0.125 (0.138)	0.129 (0.143)
$n = -3$	0.113 (0.126)	0.124 (0.138)	0.130 (0.144)	0.134 (0.150)
$n = -4$	0.116 (0.131)	0.127 (0.143)	0.133 (0.150)	0.138 (0.220)

Table 2.5: Parameters $C_{n,n'}$ in the QH state with the filling factor $\nu = 6$ in the model with static screening. The values in parentheses are obtained in an approximation with the wavefunction renormalization effects included, but all QHF and MC order parameters neglected.

$C_{n,n'}$	$n' = 2$	$n' = 3$	$n' = 4$
$n = 1$	0.054 (0.031)	0.066 (0.041)	0.073 (0.047)
$n = 0$	0.088 (0.060)	0.100 (0.070)	0.107 (0.076)
$n = -1$	0.103 (0.089)	0.115 (0.100)	0.122 (0.105)
$n = -2$	0.109 (0.121)	0.121 (0.131)	0.128 (0.136)
$n = -3$	0.113 (0.131)	0.125 (0.141)	0.132 (0.147)
$n = -4$	0.116 (0.136)	0.128 (0.147)	0.135 (0.152)

of nonvanishing QHF and MC order parameters. For comparison, in parenthesis we also show the results for the same parameters in the approximation with the QHF and MC order parameters neglected. It appears that the role of such parameters is not negligible.

2.3.2 Quantum Hall States in Strong Magnetic Field

In this subsection, we study QH states with different integer filling factors at zero temperature. We will start by first considering the states associated with the $n = 0$ and $n = 1$ Landau levels. We will show, in particular, that the qualitative features of the phase diagram obtained in Ref. [89] remain qualitatively the same after the inclusion of the static screening. At the same time, the quantitative changes will be substantial.

As stated earlier, the gap equations (2.19) through (2.22) allow a large number of solutions with different types of Haldane/Dirac masses ($\Delta_n, \tilde{\Delta}_n$) and chemical potentials ($\mu_n, \tilde{\mu}_n$). In order to identify the true ground state among them, we compare their free energies. An example of the free energies of the several lowest energy solutions at $n = 0$ Landau level is shown in Fig. 2.3. [For the explicit expression of the free energy, see Appendix C in Ref. [89].] In the model at hand, the choice of the corresponding states is strongly affected by the Zeeman interaction energy, which is one of the main factors in driving the vacuum alignment. Taking into account that there may exist a large number of other symmetry breaking effects, e.g., various on-site repulsion interaction terms [122], the actual nature of the ground states should be accepted with caution. Nevertheless, the study below is quite informative: it reveals the quantitative effects of the static screening and role of the running of the dynamical parameters in the QH regime of graphene.

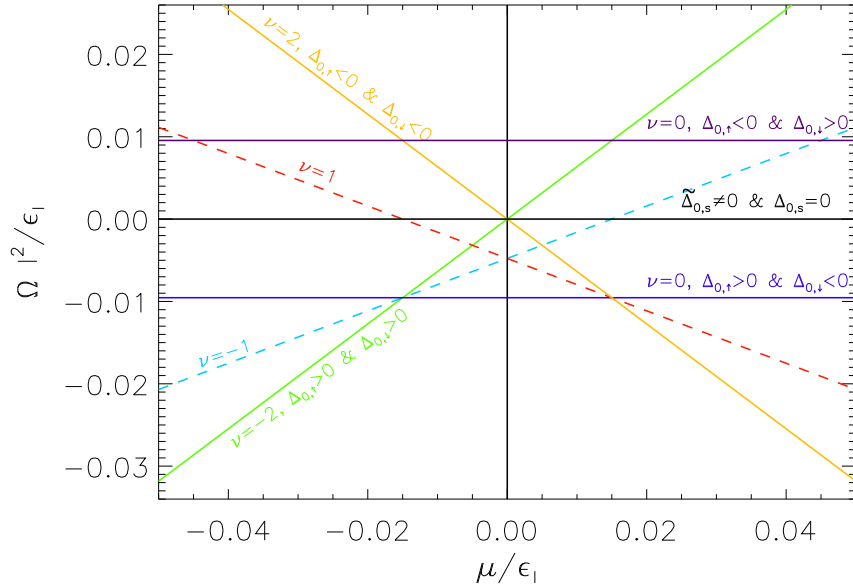


Figure 2.3: The free energies of several lowest energy solutions at $n = 0$ Landau level.

From the symmetry viewpoint, there are two types of Dirac masses and chemical

potentials for quasiparticles of each spin orientation, $s = \downarrow, \uparrow$. The parameters of the first type (i.e., Δ_n and μ_n) are singlets, while the parameters of the second type (i.e., $\tilde{\Delta}_n$ and $\tilde{\mu}_n$) are triplets with respect to the flavor $U_s(2)$ subgroups. It is natural to expect that the states with different symmetry properties compete. However, it should be emphasized that the corresponding competition is not necessarily between the MC and QHF scenarios because both types of order parameters may belong to the same representations of the flavor symmetry. In fact, as our results show, the two types of order parameters generically coexist in all QH states. (This was also emphasized in Ref. [89].)

From general considerations based on the structure of the Landau levels, it is expected that there exist at least four different classes of QH states with the following series of filling factors: (i) $\nu = 4n + 2$, (ii) $\nu = 4n$ (with a possible exclusion of the rather unique $\nu = 0$ state in a class of its own), (iii) $\nu = 4n - 1$, and (iv) $\nu = 4n + 1$.

The first series of states with the filling factors $\nu = 4n + 2$ describes the “normal” QH states with the complete filling of the (nearly) degenerate Landau levels. They do not have or require symmetry breaking and are resolved even at relatively weak magnetic fields. The simplest realization of the $\nu = 4n$ states could be provided by a dynamically enhanced Zeeman splitting of Landau levels. In this case, the ground states have $U_{\downarrow}(2) \times U_{\uparrow}(2)$ symmetry. While this also the prediction of the model at hand, we should emphasize that other realizations of the $\nu = 4n$ QH states are possible. In fact, the $\nu = 0$ state, which formally belongs to this series, is likely to have a different origin [137]. The remaining two series of states with the filling factors $\nu = 4n \pm 1$ are less controversial. They require spontaneous symmetry breaking at least down to $U_{\downarrow}(1) \times U_{\uparrow}(2)$ or $U_{\downarrow}(2) \times U_{\uparrow}(1)$.

In our analysis at sufficiently small values of the chemical potential, we find a number of different solutions, associated with the lowest Landau level ($n = 0$) and integer filling factors $\nu = 0, \pm 1, \pm 2$. The order of appearance and the competition of different types of solutions appear to be the same as in Ref. [89], see Figure 3 there. By taking into account

Table 2.6: Gap parameters for the solutions when the Fermi energy close to the lowest Landau level. The solutions for $n_{max} = 50$ with screening effects considered are compared with the solutions in [89] for $n_{max} = 5$ neglecting screening effects in the parentheses.

ν	$\tilde{\Delta}_{0,\uparrow}^{eff}$	$\tilde{\Delta}_{0,\downarrow}^{eff}$	$\Delta_{0,\uparrow}$	$\Delta_{0,\downarrow}$
-2	0.000 (0.000)	0.000 (0.000)	0.082 (0.227)	0.082 (0.227)
-1	0.000 (0.000)	0.082 (0.227)	0.082 (0.227)	0.000 (0.000)
0	0.000 (0.000)	0.000 (0.000)	0.082 (0.227)	-0.082 (-0.227)
1	0.082 (0.227)	0.000 (0.000)	0.000 (0.000)	-0.082 (-0.227)
2	0.000 (0.000)	0.000 (0.000)	-0.082 (-0.227)	-0.082 (-0.227)

the screening of the Coulomb interaction, however, we find that the actual values of dynamical parameters change substantially. The corresponding results are listed in Table 2.6. For comparison, in parenthesis we also list the previous results in the model without screening [89]. As we see, the effect of screening is to suppress the relevant dynamical parameters by about a factor of 3. The same is true for the magnitude of the energy gaps in the states with the filling factors $\nu = 0, \pm 1$.

The analysis of the QH states, associated with the $n = 1$ Landau level, is done in the same way. Here again, the order of appearance and the competition of different types of solutions appear to be exactly the same as in Ref. [89], see Figure 4 there. The actual values of the dynamical parameters in the model with screening are listed in Table 2.7. The corresponding results in the model without screening are given in parenthesis. By comparing the two sets of data, we find that screening leads to a suppression of the relevant dynamical parameters by a factor of 3 to 5.

Because of the long-range nature of the Coulomb interaction, the dynamical QHF ($\mu_n, \tilde{\mu}_n$) and MC ($\Delta_n, \tilde{\Delta}_n$) order parameters are nontrivial functions of the Landau level index n . The corresponding “running” of the dynamical parameters is an important feature

of the model at hand. From theoretical viewpoint, it is essential to provide a realistic description of the low-energy dynamics in graphene, where the role of order parameters diminishes with increasing the quasiparticle energy. This is in contrast to a common mean-field analysis of models with point-like interactions, where the order parameters affect either (i) only the nearest filled Landau level or (ii) all levels in the same way.

Our numerical results for the running dynamical parameters $f_{n,s}$, $\mu_{n,s}$, $\Delta_{n,s}$, $\tilde{\mu}_{n,s}$, and $\tilde{\Delta}_{n,s}$ are shown in Fig. 2.4 through Fig. 2.7. In line with our discussion of the four different classes of QH states with different filling factors, we show the results for $\nu = 4n + 2$, $\nu = 4n$, $\nu = 4n - 1$, and $\nu = 4n + 1$ states in separate panels. They are characterized by different ground state symmetries. In order to avoid overcrowding the figures, we showed the results only for a few states in the series, that correspond to partially or fully filled $n = 0$, $n = 1$, and $n = 2$ Landau levels. It is natural to expect that the other states in the same series, associated with filling of higher Landau levels, share essentially the same qualitative features.

In Figs. 2.4 – 2.7, the results for the spin-up and spin-down quasiparticle states are represented by the same types of filled and unfilled symbols, respectively. The universal property of all dynamical parameters is that their values approach the free model limit at large n . Additionally, we see that often the running parameters acquire their largest absolute values in the Landau levels near the Fermi energy. These features were expected, of course.

The results for the wavefunction renormalization parameters $f_{n,s}$ as functions of n are shown in Fig. 2.4. As we see, the results are qualitatively the same for all four different classes of QH states. This suggests that the wavefunction renormalization parameters and, thus, the renormalized Fermi velocities are largely determined by the long-range (screened) Coulomb interaction and not very sensitive to the effects of the dynamically generated symmetry breaking terms.

In the states with the even filling factors $\nu = 4n$ and $\nu = 4n + 2$, only the singlet order parameters $\mu_{n,\downarrow} - \mu_{n,\uparrow}$ and $\Delta_{n,s}$ are generated. In both cases, the dynamical parameters have comparable magnitudes and the ground state symmetry is $U_{\uparrow}(2) \times U_{\downarrow}(2)$. While the role of nontrivial $\mu_{n,\downarrow} - \mu_{n,\uparrow}$ and $\Delta_{n,s}$ is critical in the series of states with the filling factors $\nu = 4n$, it is not the case for the states with $\nu = 4n + 2$. Indeed, in the $\nu = 4n$ states, it is the singlet order parameters that determine the energy gaps of the QH states. They are critical because the corresponding dynamical energy gaps are generated to be much larger

Table 2.7: Gap parameters for the solutions when the Fermi energy close to the Landau level at $n = 1$. The solutions for $n_{\max} = 50$ with screening effects considered are compared with the solutions in [89] for $n_{\max} = 5$ neglecting screening effects in the parentheses.

	spin	$\tilde{\Delta}_0^{eff}$	Δ_0	f_1	$\mu_1 - \mu_s$	Δ_1	$\tilde{\Delta}_1$	f_2	$\mu_2 - \mu_s$	Δ_2	$\tilde{\Delta}_2$
$\nu = 3$	$\uparrow$	0.000 (0.000)	-0.082 (-0.227)	1.078 (1.143)	0.013 (0.053)	-0.014 (-0.068)	0.000 (0.000)	1.065 (1.112)	0.009 (0.040)	-0.010 (-0.052)	0.000 (0.000)
	$\downarrow$	-0.013 (-0.051)	-0.095 (-0.278)	1.058 (1.105)	0.050 (0.148)	-0.010 (-0.049)	0.004 (0.018)	1.062 (1.102)	0.022 (0.091)	-0.009 (-0.046)	0.000 (0.006)
$\nu = 4$	$\uparrow$	0.000 (0.000)	-0.082 (-0.227)	1.078 (1.143)	0.013 (0.053)	-0.014 (-0.068)	0.000 (0.000)	1.065 (1.112)	0.009 (0.040)	-0.010 (-0.052)	0.000 (0.000)
	$\downarrow$	0.000 (0.000)	-0.108 (-0.330)	1.038 (1.066)	0.087 (0.243)	-0.006 (-0.031)	0.000 (0.000)	1.059 (1.093)	0.034 (0.143)	-0.009 (-0.039)	0.000 (0.000)
$\nu = 5$	$\uparrow$	-0.013 (-0.051)	-0.095 (-0.278)	1.058 (1.105)	0.049 (0.148)	-0.010 (-0.049)	0.004 (0.018)	1.062 (1.102)	0.021 (0.091)	-0.009 (-0.046)	0.000 (0.006)
	$\downarrow$	0.000 (0.000)	-0.108 (-0.278)	1.038 (1.066)	0.087 (0.245)	-0.006 (-0.031)	0.000 (0.000)	1.059 (1.093)	0.034 (0.143)	-0.009 (-0.039)	0.000 (0.000)
$\nu = 6$	$\uparrow$	0.000 (0.000)	-0.108 (-0.330)	1.038 (1.066)	0.087 (0.243)	-0.006 (-0.031)	0.000 (0.000)	1.059 (1.093)	0.034 (0.142)	-0.009 (-0.039)	0.000 (0.000)
	$\downarrow$	0.000 (0.000)	-0.108 (-0.331)	1.038 (1.066)	0.087 (0.245)	-0.006 (-0.031)	0.000 (0.000)	1.059 (1.093)	0.034 (0.143)	-0.009 (-0.039)	0.000 (0.000)

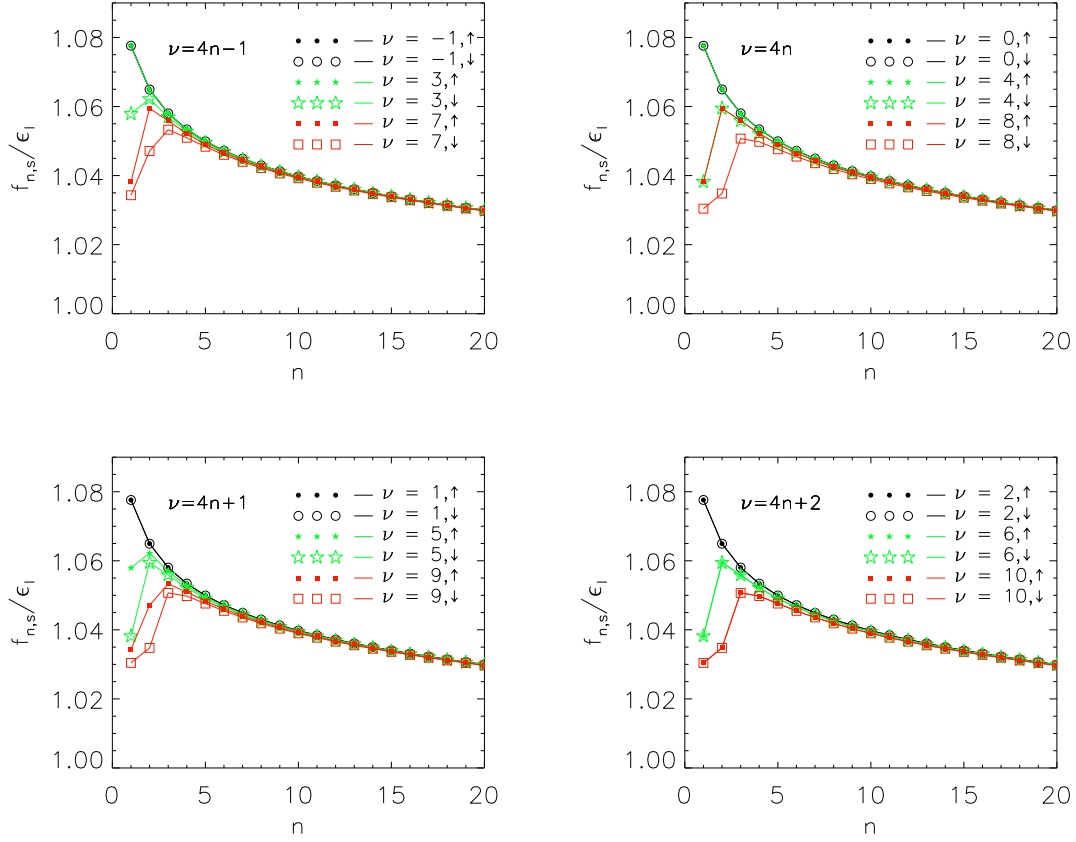


Figure 2.4: The wavefunction renormalization parameters $f_{n,s}$ as functions of the Landau level index n for four different types of QH states with filling factors $\nu = 4n - 1$, $\nu = 4n$, $\nu = 4n + 1$, and $\nu = 4n + 2$.

than the bare Zeeman splitting. This is in contrast to the $\nu = 4n + 2$ states, which are characterized by rather large gaps (of order ϵ_I) between Landau levels, which cannot be affected substantially by relatively small corrections due to $\mu_{n,\downarrow} - \mu_{n,\uparrow}$ and $\Delta_{n,s}$.

Our results for the $\nu = 4n$, associated with higher Landau levels, appear to be in qualitative agreement with the experimental results reported in Refs. [128, 130]. A spin polarized nature of the corresponding states is supported by the observed increase of the activation gaps as functions of the in-plane component of the magnetic field.

In agreement with our general symmetry arguments, the triplet order parameters

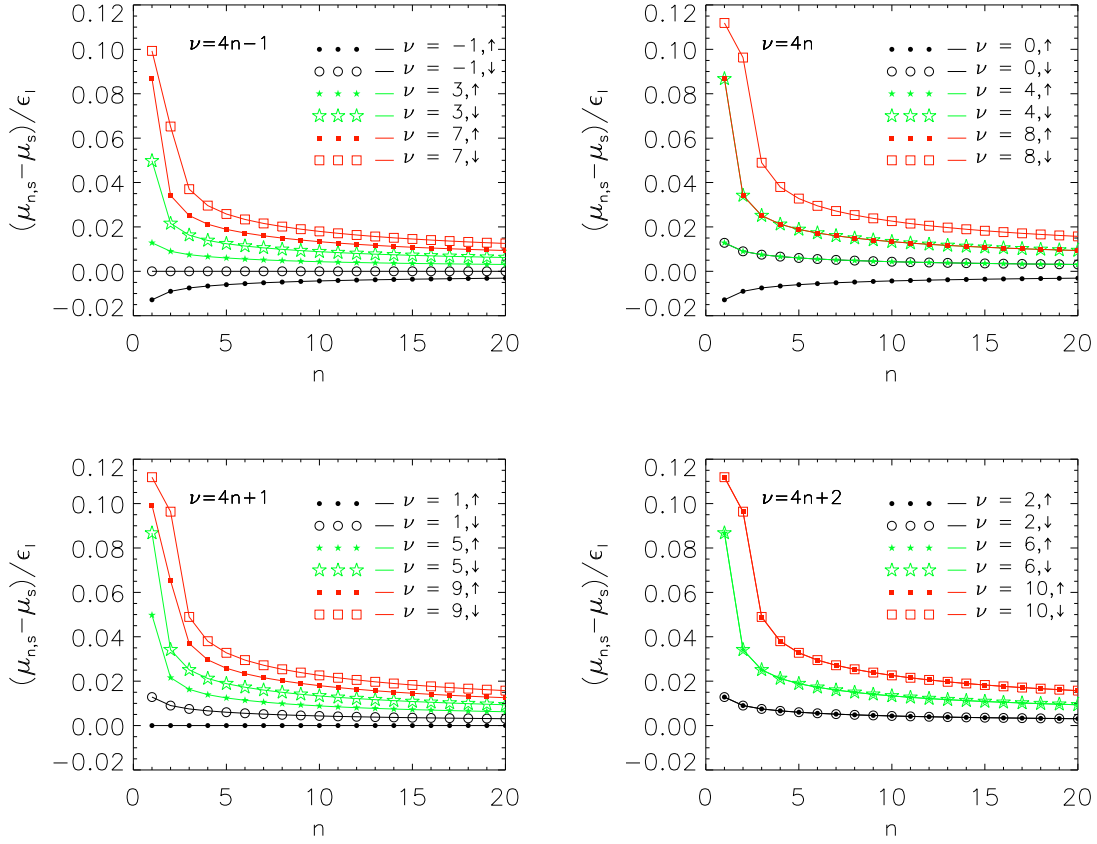


Figure 2.5: The chemical potential differences $\mu_{n,s} - \mu_s$ as functions of the Landau level index n for four different types of QH states with filling factors $\nu = 4n - 1$, $\nu = 4n$, $\nu = 4n + 1$, and $\nu = 4n + 2$.

$\tilde{\mu}_{n,s}$ and $\tilde{\Delta}_{n,s}$ are generated only in the states with the odd filling factors $\nu = 4n \pm 1$, see Fig. 2.7. We should note, however, that in both types of states the singlet order parameters $\mu_{n,\downarrow} - \mu_{n,\uparrow}$ and $\Delta_{n,s}$ are generated as well, see Figs. 2.5 and 3.2. This is not surprising since they do not break any additional symmetries. The qualitative difference between the states with $\nu = 4n - 1$ and $\nu = 4n + 1$ is that the triplet order parameters are generated either exclusively for the spin-down quasiparticles ($\nu = 4n - 1$) or exclusively for the spin-up quasiparticles ($\nu = 4n + 1$). It is not immediately clear whether these results are in perfect agreement with the experimental data in Ref. [128]. Our theoretical model appears

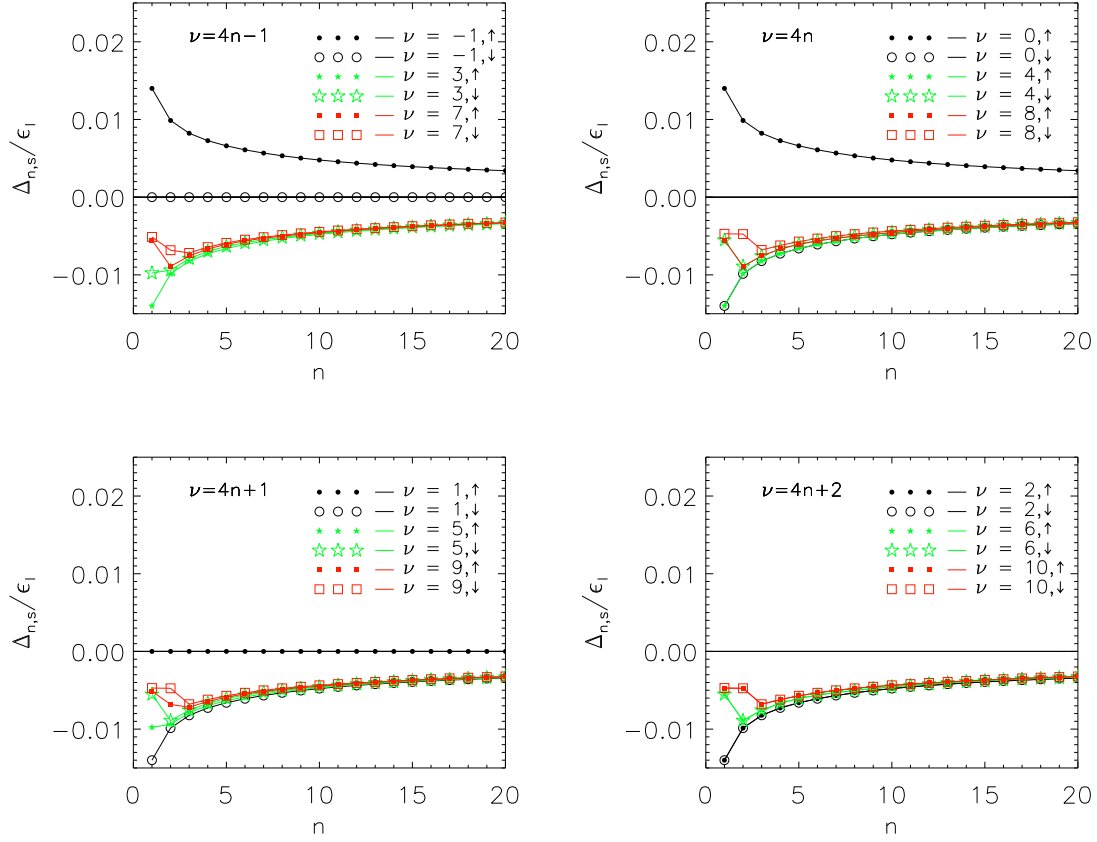


Figure 2.6: The singlet Haldane masses $\Delta_{n,s}$ as functions of the Landau level index n for four different types of QH states with filling factors $\nu = 4n - 1$, $\nu = 4n$, $\nu = 4n + 1$, and $\nu = 4n + 2$.

to capture some of the key qualitative features of the experimental data. For example, the corresponding states are characterized by a spontaneous breakdown of the flavor symmetry and the gaps are not particularly sensitive to the Zeeman energy. On the other hand, the exact symmetry breaking pattern in the observed states with filling factors $\nu = 4n \pm 1$ is not clear. In theory, the ground state symmetry is either $U_{\downarrow}(1) \times U_{\uparrow}(2)$ or $U_{\downarrow}(2) \times U_{\uparrow}(1)$. While this does not contradict the measurements, the actual symmetry could in principle be lower, i.e., $U_{\downarrow}(1) \times U_{\uparrow}(1)$. One way to resolve the issue unambiguously is to investigate the spectrum of quasiparticles in detail, e.g., by studying systematically all allowed optical

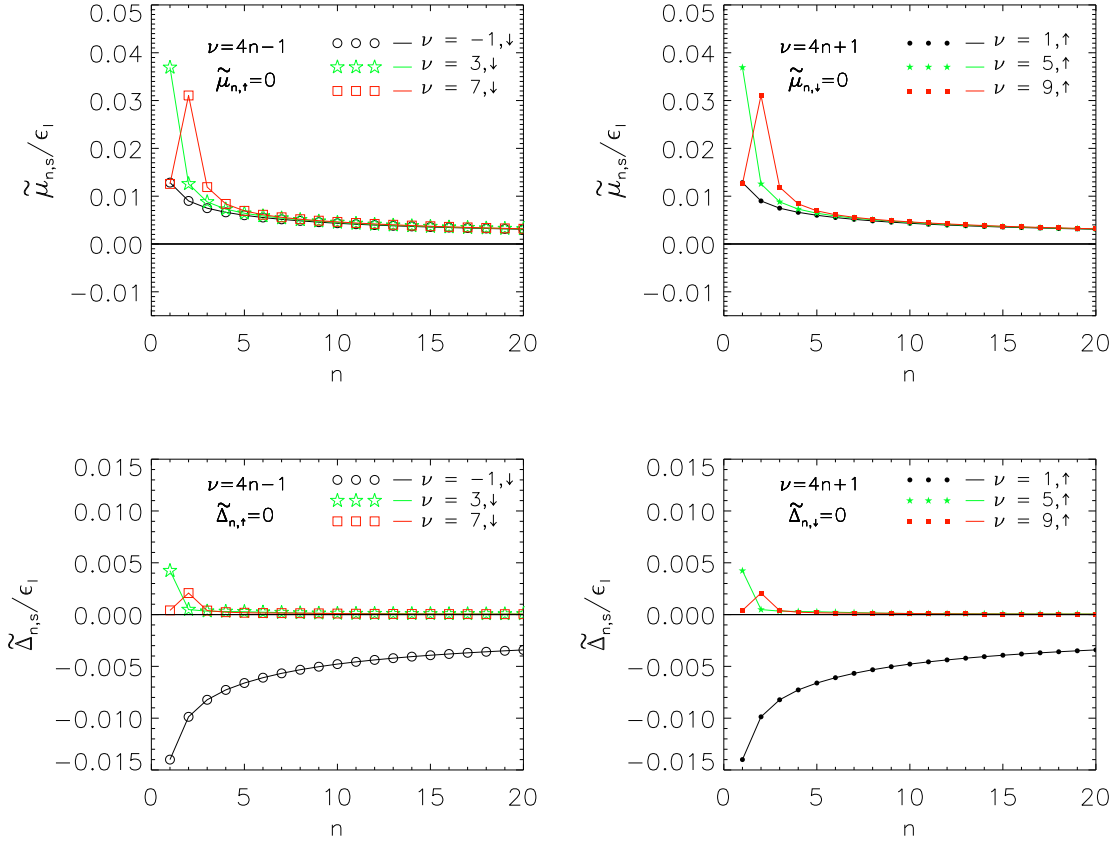


Figure 2.7: The triplet chemical potentials $\tilde{\mu}_{n,s}$ and the Dirac masses $\tilde{\Delta}_{n,s}$ as functions of the Landau level index n for the QH states with the filling factors $\nu = 4n \pm 1$.

transitions. We hope that this will be done in future investigations.

2.3.3 Temperature Dependence of the Energy Gaps

As is clear from our discussion in the previous subsection, the QH states with the filling factors $\nu = 4n$ and $\nu = 4n \pm 1$ are characterized by the energy gaps that are largely determined by the dynamically generated QHF and MC order parameters. (Recall that the energy gaps in the $\nu = 4n + 2$ states are of the order of the Landau energy scale ϵ_l and, thus, are not affected much by the dynamical order parameters.) The corresponding parameters and, therefore, the energy gaps are expected to have a strong temperature dependence.

In this subsection, we study such a dependence in detail.

The numerical analysis of the GLLR gap equations (2.19) through (2.22) at nonzero temperature is qualitatively the same as at $T = 0$. After determining the ground states at various filling factors as a function of temperature, we can straightforwardly extract the temperature dependence of the energy gaps. For different types of the QH states, associated with various fillings of the lowest three Landau levels ($n = 0, 1, 2$), our numerical results are shown in Figs. 2.8 and 2.9. There are several universal features of these results: (i) the gaps decrease monotonically with temperature, (ii) the overall size of the gaps decreases with increasing the Landau level index n . We also find that, for a fixed n , the gap functions in the states with $\nu = 4n - 1$ and $\nu = 4n + 1$ are almost exactly the same, see Fig. 2.8.

By comparing the results in Fig. 2.9 with those in Fig. 2.8, we see that the temperature dependence of the energy gaps in the QH states with filling factors $\nu = 4n \pm 1$ is qualitatively different from that in the $\nu = 4n$ states. By taking into account the very different symmetry properties of the corresponding ground states, this is not surprising at all. The $\nu = 4n \pm 1$ states are characterized by the triplet order parameters $\tilde{\mu}_{n,s}$ and $\tilde{\Delta}_{n,s}$, which vanish in a symmetry restoring phase transition at the critical temperature T_c , see Fig. 2.8. The transition appears to be either a second-order, or a weakly first-order transition. The temperature dependence of the energy gaps in the $\nu = 4n \pm 1$ states is fitted quite well by the following function:

$$\Delta E = \Delta E^{(0)} \left[1 - \left(\frac{T}{T_c} \right)^4 \right]^{0.8}, \quad (2.27)$$

where $\Delta E^{(0)}$ is the energy gap at zero temperature and T_c is the critical temperature. Note that the approximate numerical values of the zero temperature gaps are $0.165\epsilon_l$, $0.130\epsilon_l$ and $0.112\epsilon_l$ for the $\nu = 4n \pm 1$ states, associated with the Landau levels $n = 0, 1, 2$, respectively. The corresponding approximate values of the critical temperatures are $0.043\epsilon_l$, $0.033\epsilon_l$ and $0.029\epsilon_l$, respectively.

In contrast, the $\nu = 4n$ states are characterized by the singlet order parameters $\mu_{n,\downarrow} - \mu_{n,\uparrow}$ and $\Delta_{n,s}$, which have the same symmetry as the Zeeman term and, thus, remain nonzero even at large temperatures, see Fig. 2.9. As a result, the corresponding transitions from the low-temperature $\nu = 4n$ states with a dynamically enhanced Zeeman splitting to the high-temperature states without such an enhancement are generically smooth crossovers. In the case of the $\nu = 0$ state, however, we find a sign of a small discontinuity in the temperature dependence of the energy gap around $T \simeq 0.04\epsilon_l$, suggesting a weak first-order transition. Of course, such a transition is not related to a restoration of any exact symmetry and, thus, can be viewed as accidental.

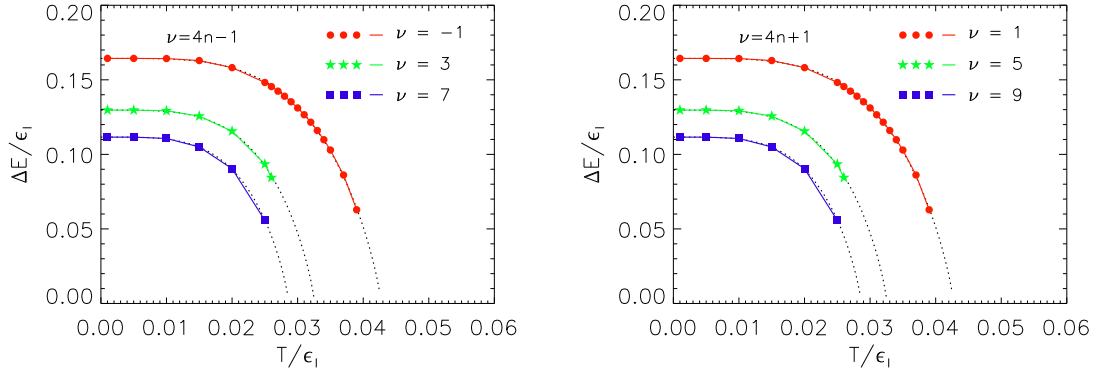


Figure 2.8: The energy gaps as functions of temperature for the QH states with the filling factors $\nu = 4n \pm 1$.

We also plot the dynamical parameters as a function of the Landau level index n at temperature $T = 0.03\epsilon_l, 0.08\epsilon_l$ in Fig. 2.10 – 2.16. The comparison of the dynamical parameters at $T = 0.03\epsilon_l, 0.08\epsilon_l$ with that at zero temperature shows that

- (i) at $T = 0.03\epsilon_l$, the dynamical parameters at $n = 4n \pm 1$ vanish at Landau level $n > 0$, which agrees with the vanishing $n = 4n \pm 1$ quantum Hall states at LL $n > 0$ as shown in Fig. 2.8. The dynamical parameters at $n = 4n, 4n + 2$ are not vanishing and show similar feature with that at zero temperature, while the values of the dynamical

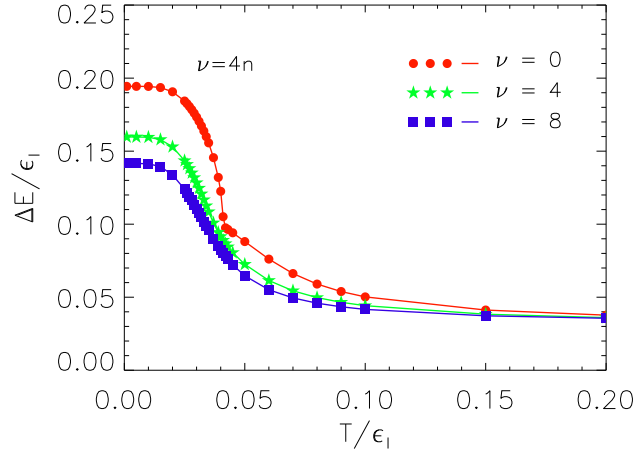


Figure 2.9: The energy gaps as functions of temperature for the QH states with the filling factors $\nu = 4n$.

parameters slightly decrease and the dynamical parameters at spin-up and spin-down slightly split rather than overlapping at nonzero temperature.

- (ii) at $T = 0.08\epsilon_l$, only dynamical parameters at $\nu = 4n, 4n + 2$ nonvanishing, and this agrees with the restored symmetry of the $\nu = 4n \pm 1$ quantum Hall states. The dynamical parameters decrease slightly with respect to the zero temperature and show spin-splitting, same as the feature shown at $T = 0.03\epsilon_l$.

2.4 Discussion

In this chapter, we utilized a highly flexible GLLR representation for the fermion propagator to describe the QH states with integer filling factors in the low-energy model of graphene with long-range Coulomb interaction. By including the static screening effects in an external magnetic field, we amended the earlier study of Ref. [89]. Our results show that the screening has a substantial suppression effect on the dynamical order parameters in all QH states with spontaneously broken symmetry. Also, the deviations of the wavefunction

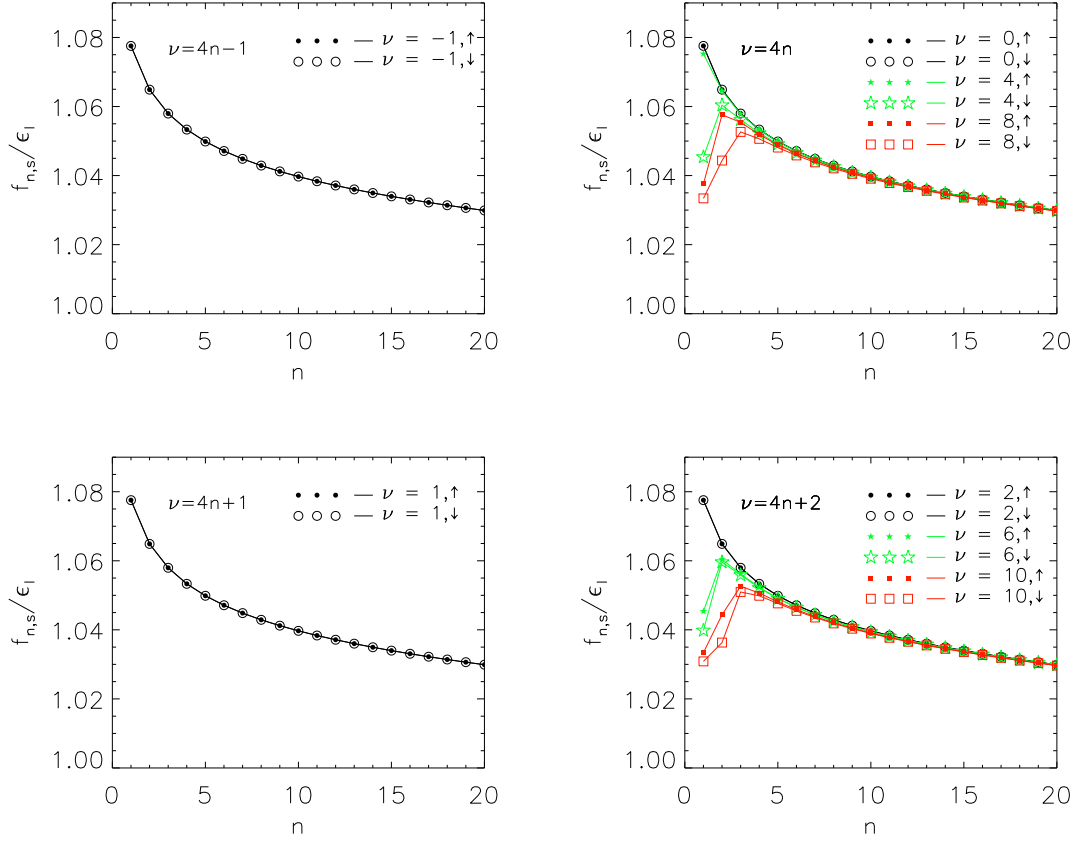


Figure 2.10: The wavefunction renormalization parameters $f_{n,s}$ as functions of the Landau level index n for four different types of QH states with filling factors $\nu = 4n - 1$, $\nu = 4n$, $\nu = 4n + 1$, and $\nu = 4n + 2$ at $T = 0.03\epsilon_l$.

renormalization from 1 and the dynamical corrections to the Fermi velocity are become noticeably suppressed.

By making use of the framework that naturally incorporates the running of the dynamical parameters with the Landau level index n , we observed that the largest absolute values of such parameters are typically obtained for the Landau level near the Fermi energy. In the limit of large n , all dynamical parameters approach the corresponding bare values. By making use of numerical methods, we studied in detail the behavior of the dynamical parameters in all four different types of the QH states with the filling factors $\nu = 4n + 2$,

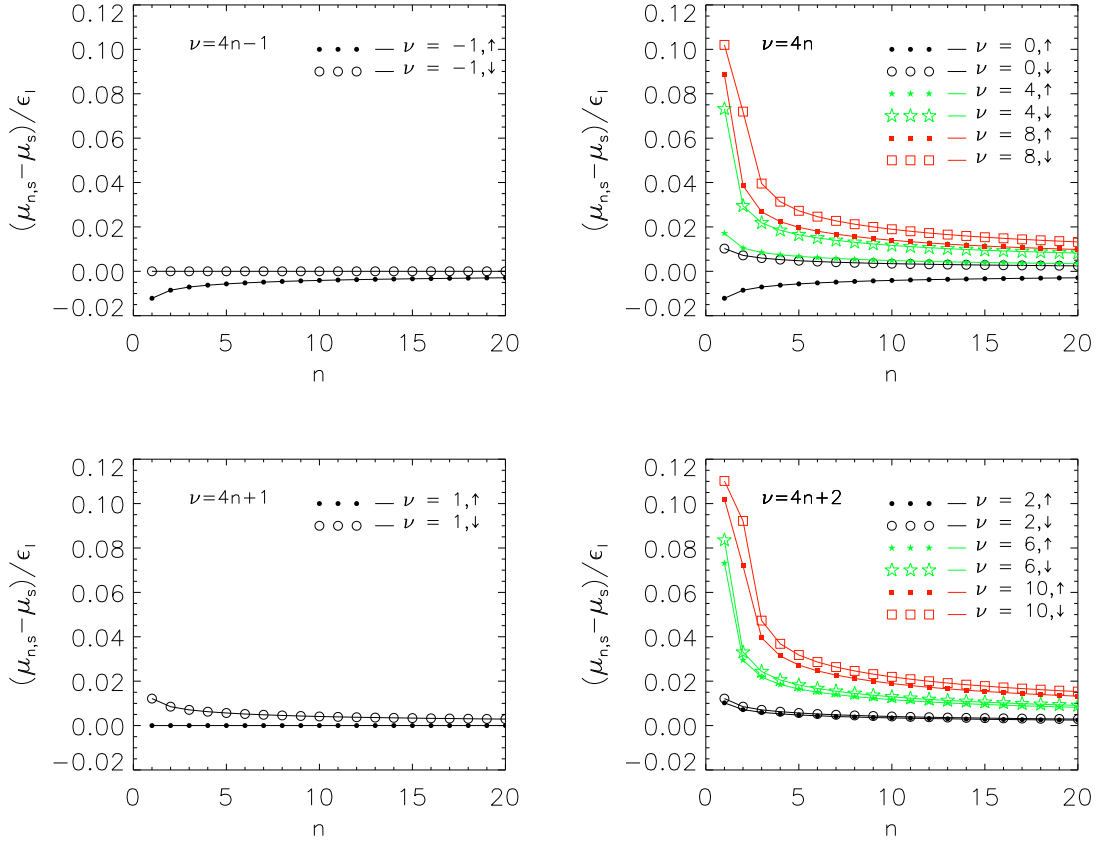


Figure 2.11: The chemical potential differences $\mu_{n,s} - \mu_s$ as functions of the Landau level index n for four different types of QH states with filling factors $\nu = 4n - 1$, $\nu = 4n$, $\nu = 4n + 1$, and $\nu = 4n + 2$ at $T = 0.03\epsilon_l$.

$\nu = 4n$, and $\nu = 4n \pm 1$ that have different symmetry properties. At weak fields, we also analyze the running of the renormalized Fermi velocity with the Landau level index n .

In the low-energy model used, the states with the filling factors $\nu = 4n + 2$ and $\nu = 4n$ have the $U_\uparrow(2) \times U_\downarrow(2)$ symmetry. They are characterized by the singlet type of the QHF and MC order parameters with respect to $U_s(2)$ for both $s = \downarrow$ and $s = \uparrow$. While the role of the corresponding singlet parameters in the $\nu = 4n + 2$ states is negligible, it is profound in the $\nu = 4n$ states, where they determine the magnitude of the dynamically enhanced Zeeman splitting. The states with the odd filling factors $\nu = 4n \pm 1$ have rather

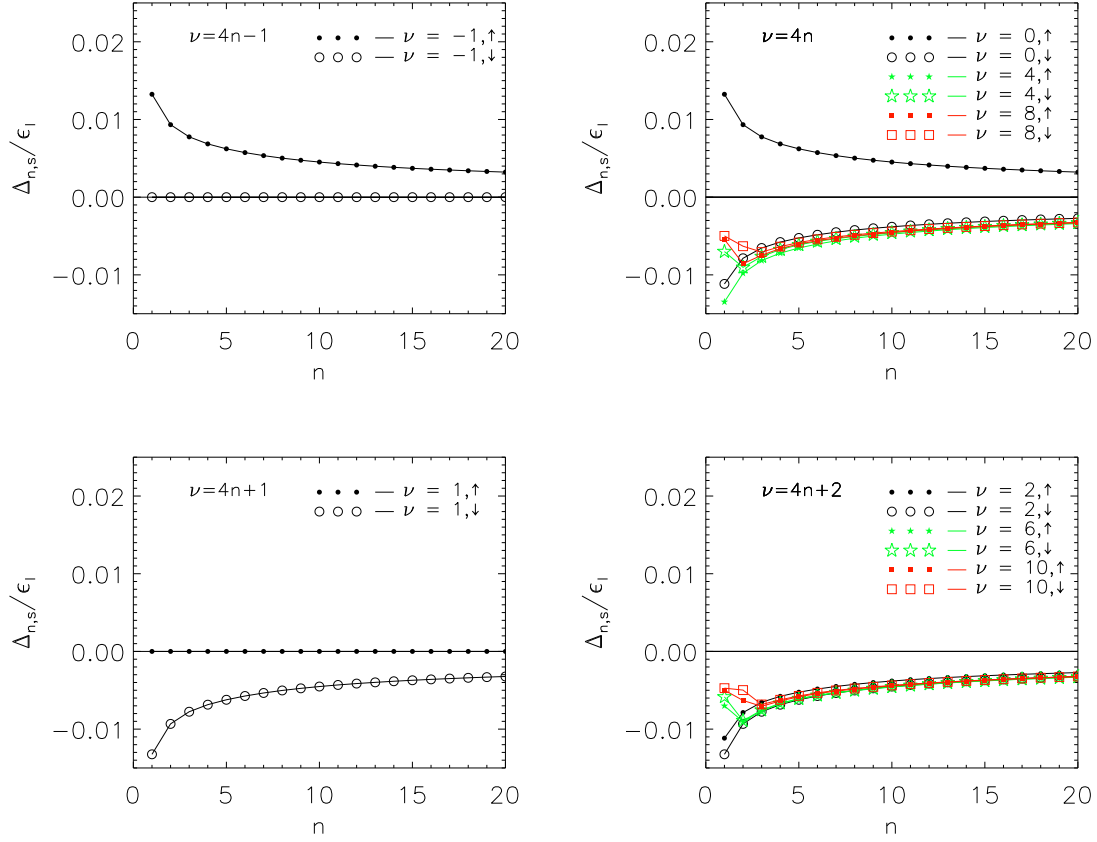


Figure 2.12: The singlet Haldane masses $\Delta_{n,s}$ as functions of the Landau level index n for four different types of QH states with filling factors $\nu = 4n - 1$, $\nu = 4n$, $\nu = 4n + 1$, and $\nu = 4n + 2$ at $T = 0.03\epsilon_l$.

different properties and are characterized by a lower ground state symmetry, i.e., either $U_\uparrow(1) \times U_\downarrow(2)$ or $U_\uparrow(2) \times U_\downarrow(1)$. In such states, in addition to the singlet order parameters, there are two types of spontaneously generated triplet parameters. The latter play a critical role in the realization of the $\nu = 4n \pm 1$ states by reducing the symmetry so as to allow the appropriate lifting of the fourfold degeneracy, as well as needed partial filling of Landau levels.

By extending the analysis to the case of nonzero temperature, we studied the energy gaps in the QH states with the filling factors $\nu = 4n$ and $\nu = 4n \pm 1$. We found that

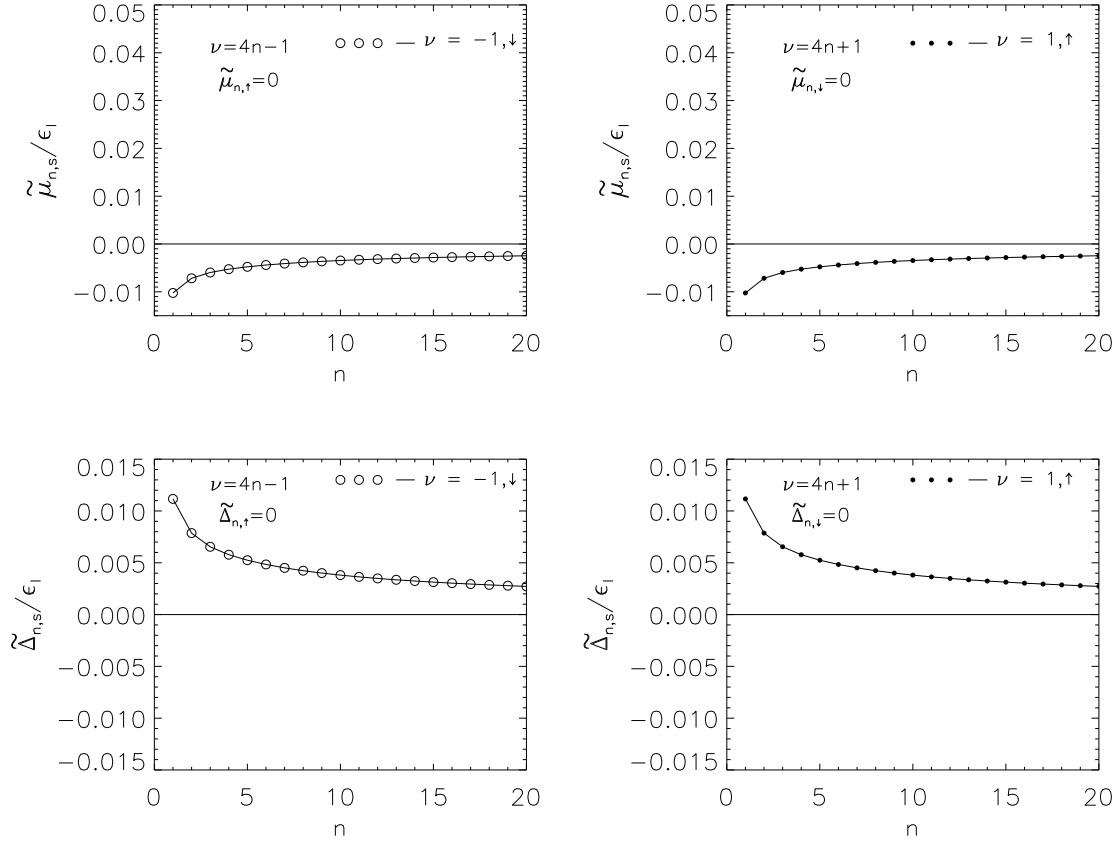


Figure 2.13: The triplet chemical potentials $\tilde{\mu}_{n,s}$ and the Dirac masses $\tilde{\Delta}_{n,s}$ as functions of the Landau level index n for the QH states with the filling factors $\nu = 4n \pm 1$ at $T = 0.03\epsilon_l$.

the symmetry breaking (triplet) order parameters, describing the $\nu = 4n \pm 1$ states, vanish at a certain critical value of temperature T_c , where a second-order or weakly first-order transition occurs. In terms of the zero temperature gap $\Delta E^{(0)}$, the results for the critical temperature are approximately given by $T_c \approx 0.26\Delta E^{(0)}$ for all $\nu = 4n \pm 1$ states associated with filling of the first few Landau levels. On the other hand, we do not detect a well-defined symmetry-restoring phase transition in the temperature dependence of the energy gaps in the $\nu = 4n$ states. Instead, we find a smooth crossover, which is consistent with the fact that the corresponding states have no real symmetry-breaking order parameters.

The results of this study unambiguously suggest that the QH states of graphene

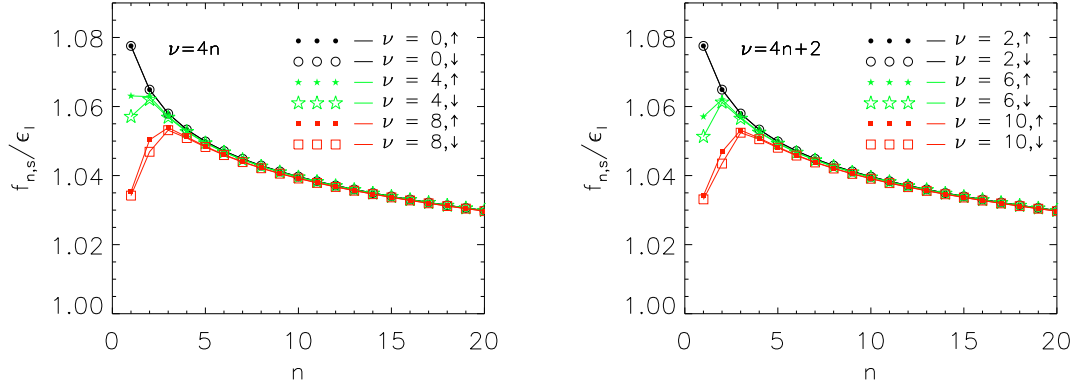


Figure 2.14: The wavefunction renormalization parameters $f_{n,s}$ as functions of the Landau level index n for four different types of QH states with filling factors $\nu = 4n - 1$, $\nu = 4n$, $\nu = 4n + 1$, and $\nu = 4n + 2$ at $T = 0.08\epsilon_l$.

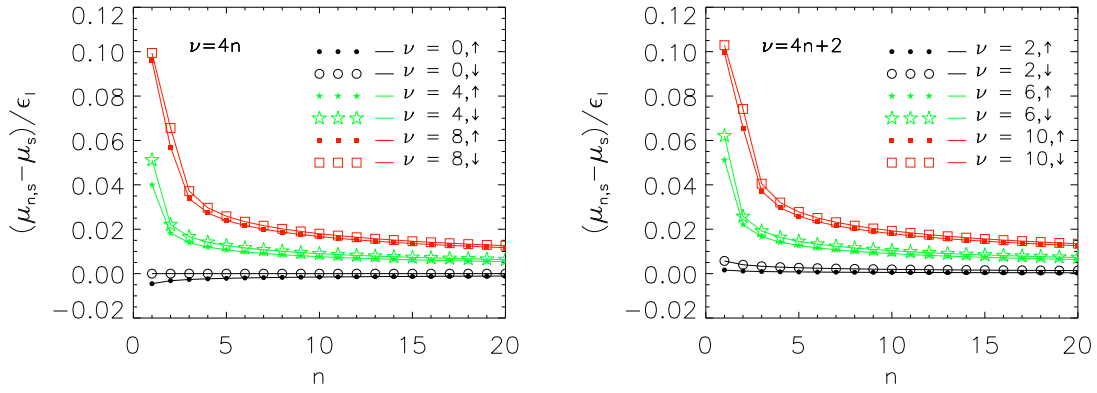


Figure 2.15: The chemical potential differences $\mu_{n,s} - \mu_s$ as functions of the Landau level index n for four different types of QH states with filling factors $\nu = 4n - 1$, $\nu = 4n$, $\nu = 4n + 1$, and $\nu = 4n + 2$ at $T = 0.08\epsilon_l$.

with various integer filling factors are generically characterized by a rather large number of dynamical parameters that have a nontrivial running with the Landau level index n . The corresponding rich structure of Landau levels in the QH regime could be probed in detail via a systematic study of transition lines in optical experiments such as those reported in Refs. [143, 144, 145]. The temperature dependence of the energy gaps obtained in

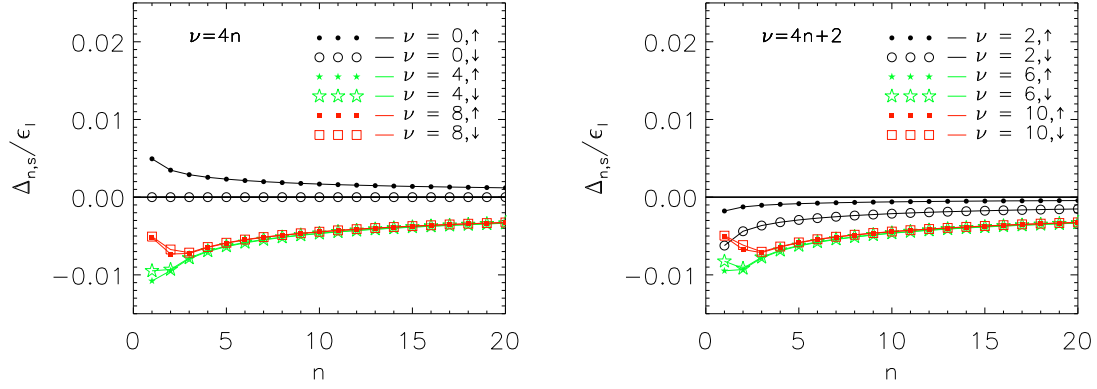


Figure 2.16: The singlet Haldane masses $\Delta_{n,s}$ as functions of the Landau level index n for four different types of QH states with filling factors $\nu = 4n - 1$, $\nu = 4n$, $\nu = 4n + 1$, and $\nu = 4n + 2$ at $T = 0.08\epsilon_l$.

this study appear to be in a qualitative agreement with the measurements of the activation energies in Ref. [70, 71, 123, 83, 84, 128, 129]. While such an agreement is encouraging, it remains to be seen how the theoretical prediction could be match also quantitatively to the experimental data.

Chapter 3

CHIRAL ASYMMETRY IN QED IN A STRONG MAGNETIC FIELD

In this chapter, we calculate the interaction induced chiral asymmetry in cold QED plasma beyond the weak-field approximation. By making use of the recently developed Landau-level representation for the fermion self-energy, the chiral shift and the parity-even chiral chemical potential function are obtained with the help of numerical methods. The results are used to quantify the chiral asymmetry of the Fermi surface in dense QED matter. Because of the weakness of the QED interactions, the value of the asymmetry appears to be rather small even in the strongest magnetic fields and at the highest stellar densities. However, the analogous asymmetry can be substantial in the case of dense quark matter.

3.1 Motivation

Nowadays the studies of chiral asymmetry in magnetized relativistic matter drew attention of researchers across diverse areas in physics. Heavy-ion collisions [8, 147, 9], compact stars [1, 3], the Early Universe [4, 5, 6], and Dirac/Weyl semimetals [18, 148] are the main physical systems where such studies are relevant. The principal role in generating a chiral asymmetry in relativistic matter with usual vector-like gauge interactions is played by an external magnetic field $\mathbf{B}$. In fact, it is the lowest Landau level (LLL) that is primarily responsible for the generation of chiral asymmetry in magnetized relativistic matter. The LLL is special because it has fermion spins completely polarized: they are directed along the magnetic field for a positive charge and opposite to the field for a negative one. For massless or ultrarelativistic particles, it is more appropriate to talk about their helicity rather than spin. Since the helicity of massless particles is the same as their chirality and the chiralities of the left- and right-handed particles are opposite, it is easy to understand how a non-dissipative axial current [149] $\mathbf{j}_5 = e\mathbf{B}\mu/(2\pi^2)$ is generated in magnetized relativistic

fermion matter at nonzero chemical potential μ [95]. This result is known in the literature as the chiral separation effect (CSE) (see, e.g., Ref. [150]) and only the lowest Landau level contributes to the axial current in the free theory [96].

The chiral magnetic effect (CME) [102, 150, 104] is in a sense a dual phenomenon to the chiral separation effect. The CME implies that the chiral asymmetry in magnetized relativistic matter, e.g., described by a non-vanishing chiral chemical potential $\mu_5 \neq 0$, causes a non-dissipative electric current $\mathbf{j} = e^2 \mathbf{B} \mu_5 / (2\pi^2)$. (See, however, the recent holographic study in Ref. [151], which points some fundamental differences between the realization of the CME and CSE.) Moreover, an interplay of the chiral separation and chiral magnetic effects gives rise to a novel type of collective gapless excitations: the chiral magnetic wave (CMW) [107]. Indeed, according to the CSE, a local fluctuation of the electric charge density induces a local fluctuation of the axial current. The resulting fluctuation of the chiral chemical potential produces a local fluctuation of the electric current via the CME. The latter in turn leads again to a local fluctuation of electric charge density and thus provides a self-sustaining mechanism for the propagation of a chiral magnetic wave. In heavy-ion collisions, such a wave leads to the quadrupole CME [100, 109]. One of the observable implications of the latter is the splitting of the elliptic flows of positive and negative pions, i.e., $v_2^{\pi^-} - v_2^{\pi^+} = r_e A$, where A is the net charge asymmetry $A = (\bar{N}^+ - \bar{N}^-) / (\bar{N}^+ + \bar{N}^-)$ and r_e is the slope parameter [109]. Such a splitting was observed by the STAR collaboration [152, 153, 154] and appears to be in agreement with theoretical predictions.

Apart from the CSE and CME, there also exist other related anomalous transport phenomena, e.g., the chiral vortical effect (CVE) [103, 155, 156, 157], the chiral electric effect (CEE) [158], and the chiral charge generation effect (CCGE) [159, 160]. In the free theory, these effects can be directly deduced from the chiral and gravitational anomalies. The chiral anomaly [92] describes the violation of the classically conserved axial symmetry

at the quantum level. It should be noted that in the presence of an external magnetic field the chiral anomaly is generated entirely on the lowest Landau level [106]. It is essential that the operator relation for the chiral anomaly is one-loop exact and cannot get any higher-order radiative corrections [92]. Since the chiral anomalous effects in the free theory are generated by the quantum anomalies, it was argued in Refs. [96, 97] that the one-loop results for the anomalous transport coefficients should be exact. One should keep in mind, however, that in order to get these anomalous transport coefficients one should calculate the ground state expectation values of the corresponding operator relations. Therefore, *a priori* there is no guarantee that interaction corrections should be absent.

The first studies of the interaction effects were done in Refs. [99, 110, 108] in the framework of the dense Nambu–Jona-Lasinio (NJL) model in a magnetic field. Using the method of Schwinger–Dyson equation for the fermion propagator, it was found that the interaction between the fermions in LLL and higher Landau levels promotes the chiral asymmetry from the LLL to all Landau levels [100]. For a magnetic field directed along the positive z -axis, dynamically generated chiral shift parameter Δ enters the effective action as the $\Delta \bar{\psi} \gamma^3 \gamma^5 \psi$ term and produces an additional dynamical contribution to the axial current. It should be emphasized that since this term does not break any symmetry in dense relativistic matter in a magnetic field, the chiral shift is already generated in the perturbation theory [99, 100]. In the NJL model the chiral shift takes a constant value independent of the momentum and the Landau level index. In the chiral limit, it determines the relative shift of the momenta in the dispersion relations of opposite chirality fermions $k^3 \rightarrow k^3 \pm \Delta$, where the momentum k^3 is directed along the magnetic field. In other words, it splits the Dirac point into two Weyl nodes separated in momentum space by 2Δ . It was proposed, therefore, that the same mechanism should take place in Dirac semimetals at nonzero charge density: in a magnetic field they transform into Weyl semimetals [161].

Direct quantum-field theoretical calculations performed in dense QED to the lead-

ing order in coupling and the external magnetic field [101] showed that the axial current in the CSE receives nontrivial radiative corrections. It was found that, in the weak-field limit, the radiative corrections to the CSE originate from the singularities at the Fermi surface. (The radiative corrections to the chiral vorticity conductivity connected with the CVE were calculated in Refs. [162, 149].) The role of the interaction effects and radiative corrections in various chiral anomalous effects in magnetized relativistic matter were recently discussed in Refs. [163, 164].

By calculating the electron self-energy in magnetized QED plasma to the leading order in the coupling constant and the external magnetic field, it was found in Ref. [112] that the chiral asymmetry of the normal ground state of the system is characterized by two distinct Dirac structures. While one of them is the chiral shift familiar from the NJL model studies, the other Dirac structure is new. It formally looks like that of the chiral chemical potential but is an odd function of the longitudinal component of the momentum (i.e., directed along the magnetic field). The origin of this new parity-even chiral structure is directly connected with the long-range character of the QED interaction. The calculations in Ref. [112] were performed in the weak magnetic field approximation, using the pseudomomentum representation. Recently, the same pseudomomentum representation was also tested in the problem of chiral symmetry breaking in QCD at zero baryon density [165].

The calculation of the fermion self-energy (as well as the chiral asymmetry functions) in the weak-field limit [112] revealed an infrared logarithmic singularity. This feature may well be an artifact of the expansion in powers of the magnetic field. It may also be related to yet another problem. As shown in Ref. [101], the weak-field result for the axial current density depends on the photon mass which is introduced as an infrared regulator. It was argued, however, that such a dependence is fictitious and is expected to go away after taking into account the nonperturbative corrections beyond the weak-field limit. As a first step in the direction of resolving the limitations of the weak-field expansion [112], in this

chapter we study the fermion self-energy and chiral asymmetry in cold magnetized QED plasma in the Landau level representation.

This chapter is organized as follows. In Sec. 3.2, we briefly introduce the model and notations. The definition of the fermion self-energy and its Landau-level representation are reviewed in Sec. 3.2.2. In the same section, we also define the chiral asymmetry functions and discuss their UV properties. The numerical algorithm for calculating the chiral asymmetry functions, as well as the main results are presented in Sec. 3.3. In Sec. 3.4, we summarize our findings and give our conclusions. In several appendices at the end, we provide some technical details and derivations used in the main text.

3.2 Model

Following closely the notation of Ref. [112], we start from the following Lagrangian density of QED in an external magnetic field:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi} \left(i\gamma^\nu \mathcal{D}_\nu + \mu\gamma^0 - m \right) \psi, \quad (3.1)$$

where $\mathcal{D}_\nu = \partial_\nu - ieA_\nu^{\text{ext}} - ieA_\nu$ is the covariant derivative, μ is the fermion chemical potential, and m is the bare fermion mass. Note that the notation is similar to that of Ref. [166], but assumes the opposite sign of the electric charge e , i.e., our e is positive. Without loss of generality, we assume that the external magnetic field $\mathbf{B}$ points in the z direction. The components of the spatial vectors, including those of the vector potential $\mathbf{A}^{\text{ext}}$, are identified with the *contravariant* components. The components of the gradient ∇ are given by $\partial_k \equiv -\partial^k$. When the explicit form of the vector potential is needed, we utilize the Landau gauge, $\mathbf{A}^{\text{ext}} = (0, xB, 0)$.

3.2.1 Fermion Propagator

In the presence of a constant magnetic field $\mathbf{B}$, part of the translational symmetry in the system is broken. This is obvious because, for one-particle states of charged fermions, the

momentum perpendicular to the magnetic field is not a good quantum number. The absence of the translational invariance is reflected in the structure of the fermion propagator [31], i.e.,

$$S(u, u') = e^{i\Phi(\mathbf{r}, \mathbf{r}')} \bar{S}(u - u'), \quad (3.2)$$

where $u = (t, \mathbf{r})$ is a space-time four-vector, $\mathbf{r} = (x, y, z)$, $\Phi(\mathbf{r}, \mathbf{r}') = -eB(x + x')(y - y')/2$ is the Schwinger phase, and $\bar{S}(u - u')$ is the translation invariant part of the propagator. The same representation is valid for the inverse fermion propagator as well [112], i.e.,

$$S^{-1}(u, u') = e^{i\Phi(\mathbf{r}, \mathbf{r}')} \bar{S}^{-1}(u - u'). \quad (3.3)$$

It should be emphasized, though, that the translation invariant part $\bar{S}^{-1}(u - u')$ is not the inverse of $\bar{S}(u - u')$.

3.2.2 Fermion Self-energy

As proposed in the previous study [112], the chiral asymmetry of the dense QED in a magnetic field is captured by the structure of the fermion self-energy. To leading order in coupling constant $\alpha = e^2/(4\pi)$, the corresponding expression for the self-energy in coordinate space reads

$$\Sigma(u, u') = -4i\pi\alpha\gamma^\mu S(u, u')\gamma^\nu D_{\mu\nu}(u - u'). \quad (3.4)$$

By taking into account the structure of the propagator in Eq. (3.2), we find that the self-energy has the same Schwinger phase factor, i.e., $\Sigma(u, u') = e^{i\Phi(\mathbf{r}, \mathbf{r}')} \bar{\Sigma}(u - u')$. After dropping the corresponding phase on both sides of Eq. (3.4) and performing the Fourier transform, we arrive at the following pseudomomentum representation for the translation invariant part of the self-energy:

$$\bar{\Sigma}(p) = -4i\pi\alpha \int \frac{dk_0 dk_3 d^2\mathbf{k}_\perp}{(2\pi)^4} \gamma^\mu \bar{S}(k) \gamma^\nu D_{\mu\nu}(k - p). \quad (3.5)$$

Here $\bar{S}(k)$ is the Fourier transform of the translation invariant part of the fermion propagator, and $D_{\mu\nu}(q)$ is the momentum space representation for the photon propagator.

In the study at hand, we are interested in properties of cold QED matter at nonzero density. Moreover, we assume that the fermion number density is large, i.e., the corresponding value of the chemical μ is much larger than other energy scales in the problem. In particular, we assume that $\mu \gg \sqrt{|eB|} \gg m$, which is a reasonable hierarchy, for example, in the case of electron plasma in magnetars. One of the most important effects associated with the nonzero density is the screening of the one-photon exchange interaction. Even at weak coupling, such screening is strong and plays an important role in the dynamics. The well known scheme that captures the corresponding effects is called the hard-dense-loop (HDL) approximation [167, 168]. The explicit form of the HDL photon propagator is given by

$$D_{\mu\nu}(q) \simeq i \left(\frac{|\mathbf{q}|}{|\mathbf{q}|^3 + \frac{\pi}{4} m_D^2 |q_4|} O_{\mu\nu}^{(mag)} + \frac{O_{\mu\nu}^{(el)}}{q_4^2 + |\mathbf{q}|^2 + m_D^2} \right), \quad (3.6)$$

where $q_4 \equiv iq_0$ and $m_D^2 = 2\alpha\mu^2/\pi$ is the Debye screening mass. In the Coulomb gauge assumed here, the Lorentz space projectors onto the electric and magnetic modes are defined as follows:

$$O_{\mu\nu}^{(mag)} = g_{\mu\nu} - u_\mu u_\nu + \frac{\mathbf{q}_\mu \mathbf{q}_\nu}{|\mathbf{q}|^2}, \quad (3.7)$$

$$O_{\mu\nu}^{(el)} = u_\mu u_\nu, \quad (3.8)$$

where $u_\mu = (1, 0, 0, 0)$.

The explicit form of the translation invariant part of the free fermion propagator is given by [169]

$$\bar{S}(k) = 2ie^{-k_\perp^2 \ell^2} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(k)}{[k_0 + \mu + i\varepsilon \operatorname{sgn}(k_0)]^2 - 2n|eB| - (k^3)^2 - m^2}, \quad (3.9)$$

where $\ell \equiv 1/\sqrt{|eB|}$ is the magnetic length and

$$\begin{aligned} D_n(k) = & \left[\gamma^0(k_0 + \mu) - \gamma^3 k^3 + m \right] \left[L_n(2k_\perp^2 \ell^2) \mathcal{P}_- - L_{n-1}(2k_\perp^2 \ell^2) \mathcal{P}_+ \right] \\ & + 2(\boldsymbol{\gamma}_\perp \cdot \mathbf{k}_\perp) L_{n-1}^1(2k_\perp^2 \ell^2). \end{aligned} \quad (3.10)$$

Here $\mathcal{P}_\pm = \left[1 \pm i \text{sgn}(eB) \gamma^1 \gamma^2\right]/2$ are spin projectors and $L_n^{(\alpha)}(x)$ are generalized Laguerre polynomials [170]. By definition, $L_{-1}(x) = 0$.

The structure of the one-loop self-energy (3.5) was discussed in detail in Ref. [112] by utilizing the Landau-level representation, recently developed in Ref. [89]. Just like the fermion propagator in Eqs. (3.9) and (3.10), the self-energy can be expanded over the Landau levels. The corresponding general form reads

$$\begin{aligned} \bar{\Sigma}(p) = & 2e^{-p_\perp^2 \ell^2} \sum_{n=0}^{\infty} (-1)^n \left\{ \left(-\gamma^0 \delta\mu_n + p^3 \gamma^3 \delta v_{3,n} - i\gamma^1 \gamma^2 \tilde{\mu}_n - \gamma^3 \gamma^5 \Delta_n \right. \right. \\ & - \left. \gamma^0 \gamma^5 \mu_{5,n} + \mathcal{M}_n \right) \left[L_n(2p_\perp^2 \ell^2) \mathcal{P}_- - L_{n-1}(2p_\perp^2 \ell^2) \mathcal{P}_+ \right] \\ & - \left. 2(\gamma_\perp \cdot \mathbf{p}_\perp) \delta v_{\perp,n} L_{n-1}^1(2p_\perp^2 \ell^2) \right\}. \end{aligned} \quad (3.11)$$

In this representation, the physical meaning of the coefficient functions $\delta\mu_n$, $\delta v_{3,n}$, etc., is obvious from their Dirac structure [112]. [Note that all these functions depend on the energy p_0 and the longitudinal momentum p_3 .]

3.2.3 Chiral Shift and Chiral Chemical Potential

In the remainder of this study, however, we will concentrate only on the two most important structures, $\Delta_n(p_3)$ and $\mu_{5,n}(p_3)$ at $p_0 = 0$, which describe the chiral asymmetry of dense QED matter. General expressions for both of these were derived in Ref. [112] by projecting the self-energy in Eq. (3.5) onto individual Landau levels, i.e.,

$$\begin{aligned} \Delta_n(p_3) = & \frac{(-1)^n}{8} \frac{l^2}{\pi} \text{sign}(eB) \int d^2 p_\perp e^{-p_\perp^2 l^2} \left[L_n(2p_\perp^2 l^2) + L_{n-1}(2p_\perp^2 l^2) \right] \text{Tr} \left[\gamma^0 \bar{\Sigma}(p) \right] \\ & - \frac{(-1)^n}{8} \frac{l^2}{\pi} \int d^2 p_\perp e^{-p_\perp^2 l^2} \left[L_n(2p_\perp^2 l^2) - L_{n-1}(2p_\perp^2 l^2) \right] \text{Tr} \left[\gamma^3 \gamma^5 \bar{\Sigma}(p) \right], \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} \mu_{5,n}(p_3) = & \frac{(-1)^n}{8} \frac{l^2}{\pi} \text{sign}(eB) \int d^2 p_\perp e^{-p_\perp^2 l^2} \left[L_n(2p_\perp^2 l^2) + L_{n-1}(2p_\perp^2 l^2) \right] \text{Tr} \left[\gamma^3 \bar{\Sigma}(p) \right] \\ & + \frac{(-1)^n}{8} \frac{l^2}{\pi} \int d^2 p_\perp e^{-p_\perp^2 l^2} \left[L_n(2p_\perp^2 l^2) - L_{n-1}(2p_\perp^2 l^2) \right] \text{Tr} \left[\gamma^0 \gamma^5 \bar{\Sigma}(p) \right], \end{aligned} \quad (3.13)$$

respectively.

Here it may be appropriate to mention that the chiral symmetry is restored in the regime of large chemical potential considered here. In this case, magnetic catalysis plays no role and the dynamical contribution to $\mathcal{M}_n$ is negligible even compared to the electron mass m . It is completely justifiable, therefore, to replace $\mathcal{M}_n$ with m in our calculations below. Moreover, the dynamical contribution to the electron mass due to the magnetic catalysis in QED is exponentially small even in the case of zero chemical potential (vacuum) [54]. This is the consequence of the smallness of the fine structure constant. While the same is not true in the QCD vacuum, the magnetic catalysis still would not play any big role in dense quark matter at large chemical potential.

Before proceeding to the numerical analysis of the chiral asymmetry functions $\Delta_n(p_3)$ and $\mu_{5,n}(p_3)$, it is instructive to discuss the implications of the well known ultraviolet (UV) divergency in the fermion self-energy function in QED [166]. In the Pauli-Villars regularization scheme, the self-energy contains the following logarithmically divergent contribution [166]:

$$\Sigma^{(div)}(p) = \frac{\alpha}{4\pi} \left[-\gamma^\nu (p_\nu + \mu \delta_\nu^0) + 4m \right] \ln \frac{\Lambda^2}{m^2}. \quad (3.14)$$

Note that the only effect of a nonzero chemical potential here is to shift $p_0 \rightarrow p_0 + \mu$ in the vacuum expression [101, 171]. Of course, the above divergency cannot be affected by the magnetic field. When projected onto Landau levels as prescribed by Eqs. (3.12) and (3.13), this result leads to the following contributions to the chiral asymmetry functions:

$$\Delta_n^{(div)}(p_3) = -\delta_n^0 \frac{\alpha(p_0 + \mu)}{8\pi} \text{sign}(eB) \ln \frac{\Lambda^2}{m^2}, \quad (3.15)$$

$$\mu_{5,n}^{(div)}(p_3) = \delta_n^0 \frac{\alpha p_3}{8\pi} \text{sign}(eB) \ln \frac{\Lambda^2}{m^2}. \quad (3.16)$$

As we see, the corresponding functions are free of the UV divergences in all, but the lowest Landau level ($n = 0$). As explained in detail in Ref. [112], the LLL ($n = 0$) is very special also because of its spin-polarized nature. As a consequence, the LLL chiral shift is

indistinguishable from the correction to the chemical potential, and the LLL axial chemical potential is indistinguishable from the correction to the longitudinal velocity. It was concluded, therefore, that the novel type of the chiral asymmetry is determined exclusively by the dynamical functions Δ_n and $\mu_{5,n}$ with $n \geq 1$. These functions are of the prime interest for us in the present chapter. In the next section, we take into account that all these functions are free from the UV divergences and study them using numerical methods.

3.3 Numerical Results

In this section we study the chiral asymmetry functions $\Delta_n(p_3)$ and $\mu_{5,n}(p_3)$ using numerical methods. To start with, we rewrite the corresponding expressions in a dimensionless form. We will measure all quantities with the dimension of energy/mass in terms of the chemical potential. For example, in the case of momenta, we will define the corresponding dimensionless quantities as follows: $x \equiv p_\perp/\mu$, $y \equiv k_\perp/\mu$, $x_3 \equiv p_3/\mu$, and $y_3 \equiv k_3/\mu$. Similarly, the dimensionless functions will be defined as follows: $\bar{\Delta}_n \equiv \Delta_n/\mu$ and $\bar{\mu}_{5,n} \equiv \mu_{5,n}/\mu$. The corresponding dimensionless forms of these chiral asymmetry functions are presented in Appendix B.

In order to analyze numerically the two chiral asymmetry functions in dense QED, we need to fix several model parameters (i.e., the strength of magnetic field, the value of the chemical potential, and the fermion mass). In principle, when using the dimensionless description, the value of the chemical potential μ may be left unspecified. Keeping in mind, however, that the value of the fermion (electron) mass has to be measured in units of μ , we will assume that the default choices of the chemical potential and the magnetic field are $\mu = 420 \text{ MeV}$ and $B = 10^{18} \text{ G}$. Then, the two dimensionless model parameters used in the

calculations will be

$$a = \frac{m}{\mu} \approx 1.22 \times 10^{-3} \frac{m}{m_e} \frac{420 \text{ MeV}}{\mu}, \quad (3.17)$$

$$b = \frac{|eB|}{\mu^2} \approx \frac{1}{30} \left(\frac{B}{10^{18} \text{ G}} \right) \left(\frac{420 \text{ MeV}}{\mu} \right)^2. \quad (3.18)$$

Note that, in agreement with the assumption made earlier, the chosen value of the magnetic field strength is rather small compared to the chemical potential scale μ^2 . By taking into account the definition of the Debye mass and the QED fine structure constant, it is also convenient to introduce the following short-hand notation for the dimensionless Debye mass:

$$d = \frac{m_D}{\mu} \equiv \sqrt{\frac{2\alpha}{\pi}} \approx 6.816 \times 10^{-2}. \quad (3.19)$$

In the final expressions for the chiral asymmetry functions, there will be a need to sum over an infinite number of Landau levels. In the numerical calculations, however, the sums will be truncated at $n_{\max} = 200$.

As is clear from the explicit expressions for the functions $\bar{\Delta}_n$ and $\bar{\mu}_{5,n}$ in Appendix B[see Eqs. (2.25) and (2.26), respectively], the numerical calculation for each of them reduces to performing four integrations: three integrations over the dimensionless momenta x , y and y_3 , and one over the angular coordinate ϕ . Taking into account that $\bar{\Delta}_n$ and $\bar{\mu}_{5,n}$ also have an additional functional dependence on the longitudinal momentum $x_3 \equiv p_3/\mu$, the corresponding task becomes rather expensive numerically.

3.3.1 Monte Carlo Importance Sampling

Before proceeding to the actual results, let us briefly describe the algorithm that we use in the calculations. The four-dimensional integrals that define the chiral asymmetry functions have the following schematic form:

$$I = \int_0^\infty dx \int_0^\infty dy \int_{-\infty}^\infty dy_3 \int_0^{2\pi} \frac{d\phi}{2\pi} f(x, y, y_3, \phi). \quad (3.20)$$

In order to calculate such an integral, we will make use of the importance sampling Monte Carlo method [172]. In such a framework, the result of the integration is approximated by a weighted sum of contributions calculated at a large number of random points in the phase space, i.e.,

$$I_N = \frac{1}{N} \sum_{i=1}^N \frac{f(x^i, y^i, y_3^i, 2\pi u^i)}{P_1(x^i)P_2(y^i)P_3(y_3^i)}. \quad (3.21)$$

Having limited information about the angular dependence of the integrand function, we use the simplest uniform distribution of the random variable $u = \phi/(2\pi)$ on the interval from 0 to 1. The other three random number variables are distributed with the probability density functions $P_1(x)$, $P_2(y)$, and $P_3(y_3)$, respectively. The specific choice of these functions will be explained momentarily. First, however, let us note that the statistical error of the Monte Carlo integration is given by the following estimator [172]:

$$\varepsilon = \frac{1}{\sqrt{N}} \sqrt{\frac{1}{N} \sum_{i=1}^N \left(\frac{f(x^i, y^i, y_3^i, 2\pi u^i)}{P_1(x^i)P_2(y^i)P_3(y_3^i)} \right)^2 - I_N^2}. \quad (3.22)$$

With increasing N , the Monte Carlo estimate I_N may converge rather slowly to the true value I . This is where importance sampling can improve the situation. The key observation is that, for a fixed number of sampling points N , the result for the above statistical error depends on the random number distributions used. The error becomes smaller when the corresponding probability densities approximate closer the integrand function itself. The same condition determines when the fastest convergence of the Monte Carlo method is achieved.

While testing our numerical algorithm, we tried performing calculations with several different types of the functional forms for the random number distributions (e.g., Gaussian and gamma distributions) and examined a number of different choices of their parameters. In such tests, the least value of the estimator (3.22) was used as an indicator of the integration effectiveness. This allowed us to make an optimal choice of the random number distributions.

In the case of the perpendicular momenta variables x and y , we ended up using the following gamma distribution:

$$P(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad (3.23)$$

where the shape and scale parameters are $\alpha = 1$ and $\beta = 1.5$, respectively. Such a distribution appears indeed appropriate in the case of the integrand function that decreases exponentially with the perpendicular momenta. In order to generate gamma-distributed random numbers, we used the *FORTRAN* code written by Richard Chandler [173].

As a quick examination of the explicit expressions for $\bar{\Delta}_n$ and $\bar{\mu}_{5,n}$ in Appendix B reveals, the dependence of their integrands on the longitudinal momentum y_3 is quite different from the dependence on x and y . In particular, they have a power-law instead of exponential behavior at large y_3 . Fig. 3.1 shows the integrands of Δ_n as a function of y_3 with all other variables at fixed values, $x = y = 0.5$, $\phi = 0.2\pi$, and with different sampling functions applied, Gaussian distribution (blue line), 4th-order power-law (purple line), Cauchy distribution (red line), and Gamma distribution (light blue). It is demonstrated that the Cauchy distribution follows the original integrand better than other three functions. The sharp exponential drop of the Gaussian distribution at the wing of the integrand induce large errors in the sampling, similarly for other two functions. Because of this, neither Gaussian nor gamma distributions were able to provide a quick convergence of the Monte Carlo integration. Instead, we used the Cauchy distribution with the following power-law probability density function for generating the longitudinal momentum variable y_3 :

$$P_3(y_3) = \frac{1}{\pi} \frac{1}{y_3^2 + 1}. \quad (3.24)$$

The random numbers with such a probability density are generated using the quantile function $y_3 = \tan \left[\frac{\pi}{2} (2p - 1) \right]$, where p is a random number with the uniform distribution on the interval between 0 and 1.

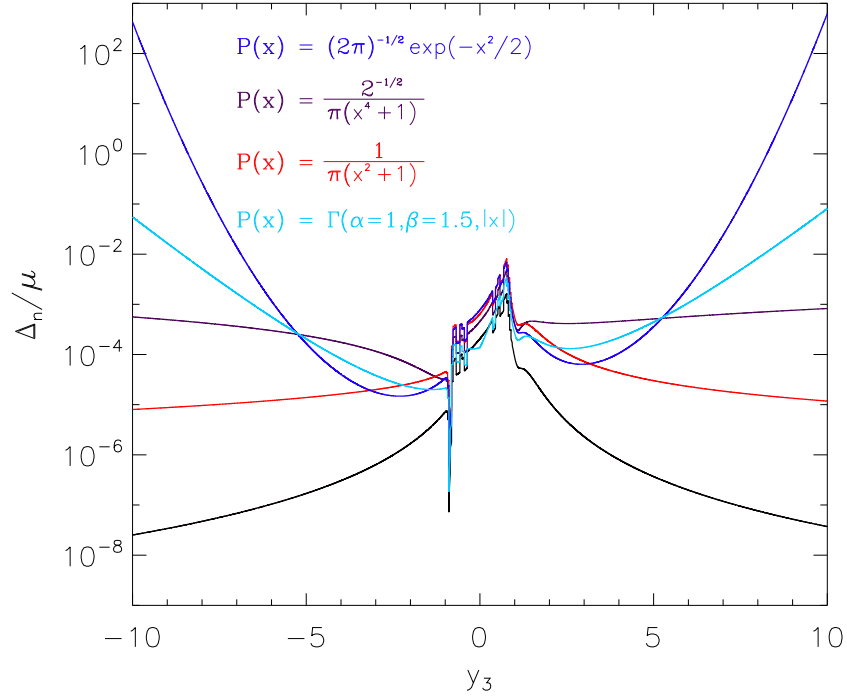


Figure 3.1: The sampling of the integrand of Δ_n as a function of y_3 with all other variables at fixed values, $x = y = 0.5$, $\phi = 0.2\pi$. Integrands with different sampling functions, Gaussian (blue), 4th-order power-law (purple), Cauchy (red), and Gamma (light blue) applied are compared with the original integrand (black).

3.3.2 Chiral Shift

Our numerical results for the chiral shift are summarized in Fig. 3.2. In the left panel, we show the dependence of the chiral shift Δ_n on the longitudinal momentum $y_3 = p_3/\mu$ for several low-lying Landau levels. Since obtaining the complete functional dependence on p_3 is rather expensive numerically, we used only a moderately large number of sampling points, $N = 2 \times 10^8$ and calculated the results only for the first four lowest lying Landau levels. The common feature of the corresponding functions is the appearance of a maximum at an approximate location of the Fermi surface. In the free (weakly interacting) theory, this is determined by the following value of the longitudinal momentum: $p_3/\mu = \sqrt{1 - 2nb - a^2}$.

In agreement with this expression, the location of the maximum of the chiral shift function in the n th Landau level $\Delta_n(p_3)$ decreases with increasing n . At large values of the momentum p_3 , the chiral shift function decreases and gradually approaches zero as expected.

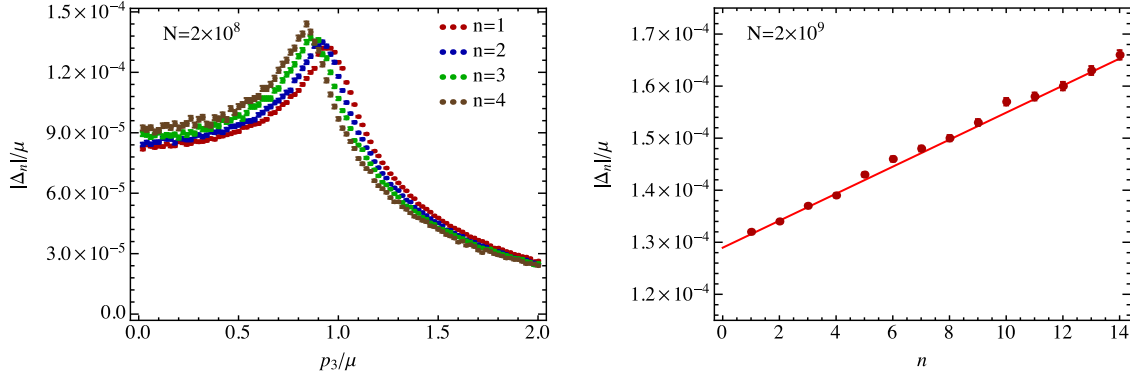


Figure 3.2: Left panel: the chiral shift Δ_n as a function of the longitudinal momentum p_3 for $n = 1$ (red), $n = 2$ (blue), $n = 3$ (green), and $n = 4$ (brown) Landau levels. Right panel: the values of the chiral shift Δ_n at the Fermi surface.

From the viewpoint of the low-energy physics, it is most important to know the chiral shift at the Fermi surface. The corresponding results are presented in the right panel of Fig. 3.2. By assumption, the location of the Fermi surface is determined by the perturbative expression, $p_3/\mu = \sqrt{1 - 2nb - a^2}$. In this calculation, we used a larger number of sampling points, $N = 2 \times 10^9$. As we see, the Fermi surface values of Δ_n grow with the Landau-level index n . (The corresponding numerical values are also given in the first column of Table 3.1.) This growth is somewhat surprising, but is in agreement with the general behavior of functions $\Delta_n(p_3)$ shown in the left panel of Fig. 3.2. The corresponding dependence on the Landau-level index can be fitted quite well by a linear function.

It is easy to check that the numerical results for the chiral shift in Fig. 3.2 are of the same order of magnitude as $\alpha|eB|/\mu$. Taking into account that Δ_n is one of the structures in the fermion self-energy, induced by a nonzero magnetic field, it is indeed quite natural that the corresponding function is proportional to the coupling constant and the magnetic

field strength. As for the chemical potential in the denominator, it is the only other relevant energy scale in the problem that can be used to render the result for Δ_n with the correct energy units. (Formally, the fermion mass is yet another energy scale, but it is unlikely to play a prominent role at the Fermi surface in the high density and strong-field limit.) The linear fit for the chiral shift at the Fermi surface is shown by the solid line in the right panel of Fig. 3.2. The corresponding function can be presented in the following form:

$$\Delta_n \simeq -\frac{\alpha|eB|}{\mu} \left(0.53 + 0.32 \frac{|eB|n}{\mu^2} \right), \quad (3.25)$$

where we took into account that the numerical results in Fig. 3.2 were obtained for the magnetic field $|eB| = \mu^2/30$ and $\alpha = 1/137$. The result in Eq. (3.25) should be contrasted with a very different parametric dependence obtained in the weak-field limit in Ref. [112], i.e., $\Delta_n \propto \alpha|eB|\mu/m^2$, which is a factor of $(\mu/m)^2$ larger. Such a large factor is quite natural in the weak-field limit, where it is an artifact of the expansion in powers of $|eB|/m^2$. In contrast, one does not expect anything like that in the case of a strong magnetic field.

3.3.3 Chiral Chemical Potential

The numerical results for the chiral chemical potential $\mu_{5,n}$ are summarized in Fig. 3.3. In the left panel, we present the chiral chemical potential in the $n = 1$ Landau level as a function of the longitudinal momentum p_3 . (The results for larger n are expected to have the same qualitative dependence on p_3 .) The red and blue points represent the results for two different numbers of sampling points, $N = 2 \times 10^8$ and $N = 2 \times 10^9$, respectively. The numerical results confirm that $\mu_{5,n}$ is an odd function of p_3 and, as such, it does not break parity. The dependence of $\mu_{5,n}$ on p_3 also reveals a pair of sharp peaks on the Fermi surface at $p_3/\mu \simeq \pm\sqrt{1-2nb-a^2}$. In the context of the low-energy physics, it is these values of $\mu_{5,n}$ on the Fermi surface that are of main importance.

The numerical results for the chiral chemical potential at the Fermi surface are shown in the right panel of Fig. 3.3. In the corresponding calculation, we again assumed

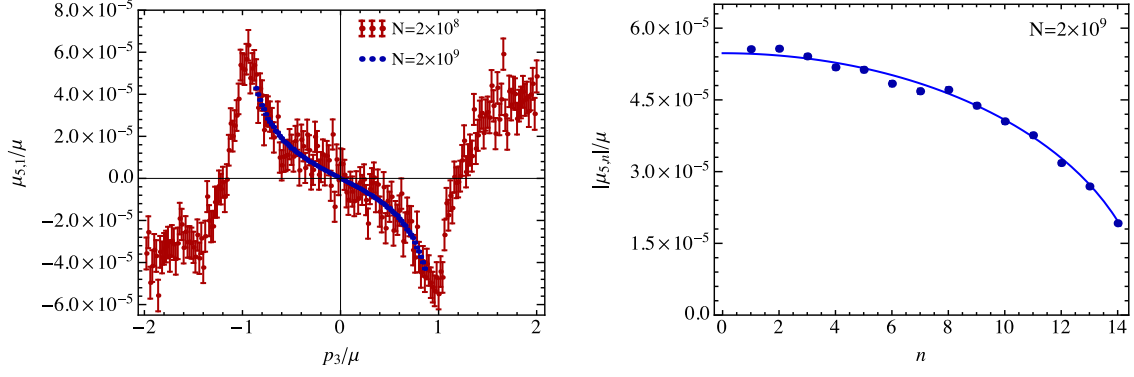


Figure 3.3: Left panel: the chiral chemical potential $\mu_{5,n}$ as a function of the longitudinal momentum p_3 for $n = 1$ Landau level. Right panel: the values of the chiral chemical potential $\mu_{5,n}$ at the Fermi surface.

that the location of the Fermi surface is determined by the perturbative expression, $p_3/\mu = \pm\sqrt{1 - 2nb - a^2}$, and used the Monte Carlo integration algorithm with $N = 2 \times 10^9$ sampling points. We find that the values of $\mu_{5,n}$ decrease with the Landau-level index n . (The corresponding numerical values are given in the second column of Table 3.1.) The order of magnitude of the obtained results is similar to those for the chiral shift function. Following the same arguments, therefore, we can assume that $\mu_{5,n}$ is also proportional to the coupling constant and the magnetic field strengths, i.e., $\mu_{5,n} \propto \alpha|eB|/\mu$. (Let us emphasize again that this dependence is quite different from the weak-field limit in Ref. [112].) In order to fit the numerical results, we could try to use a polynomial function of n . However, by following a trial and error approach instead, we found that the following simple function approximates our numerical results really well:

$$\mu_{5,n} \simeq -0.225 \frac{\alpha|eB|}{\mu} \sqrt{1 - \left(\frac{2n|eB|}{\mu^2} \right)^2}, \quad (3.26)$$

where we took into account that $|eB| = \mu^2/30$ and $\alpha = 1/137$. The corresponding function is shown by the solid line in the right panel of Fig. 3.3.

3.3.4 Chiral Asymmetry

By making use of the analytical expression for the fermion propagator with the chiral asymmetry in Appendix A, as well as the above numerical results for the chiral shift and the chiral chemical potential, we can straightforwardly determine the interaction-induced deviations of the Fermi momenta $(p_3 - p_3^{(0)})/\mu$ for the predominantly left-handed and right-handed fermions in the considered ultrarelativistic limit $\mu \gg m$. Here $p_3^{(0)}$ is the value of the Fermi momentum in the absence of the chiral asymmetry (i.e., $\Delta_n = 0$ and $\mu_{5,n} = 0$). Such deviations can be viewed as the actual measure of the chiral asymmetry at the Fermi surface. The numerical results for $(p_3 - p_3^{(0)})/\mu$ in each occupied Landau level are shown in Fig. 3.4. (The corresponding numerical values are also presented in the last two columns of Table 3.1.) This is a generalization of the analogous results in the weak-field limit, obtained in Ref. [112].

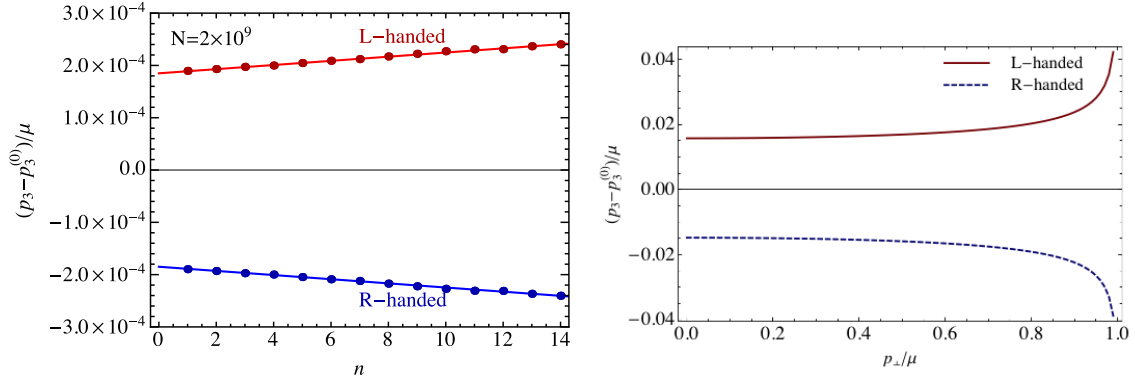


Figure 3.4: Left panel: chiral asymmetry $(p_3 - p_3^{(0)})/\mu$ of the Fermi surface for predominantly left-handed (red) and right-handed (blue) particles as a function of the Landau-level index n in this paper. Right panel: chiral asymmetry as a function of the perpendicular momentum, for predominantly left-handed and right-handed fermions in weak magnetic field [100].

We find that the results for $(p_3 - p_3^{(0)})/\mu$ in Fig. 3.4 are well approximated by linear functions of n . When written in the same form as the chiral shift and the chiral chemical

potential functions, the corresponding linear fits take the following form:

$$p_3 - p_3^{(0)} \simeq \pm \frac{\alpha |eB|}{\mu} \left(0.76 + 0.49 \frac{|eB|n}{\mu^2} \right). \quad (3.27)$$

As is easy to check, the values of these Fermi momenta shifts are of the order of 10 keV and, thus, are not very large in the context of compact stars, even though we already assumed an extremely strong value of the magnetic field, $B = 10^{18}$ G. One should keep in mind, however, that here we used the model of a dense QED plasma, whose coupling constant α is extremely small. This conclusion could change drastically in the case of dense quark matter, where the relevant coupling constant α_s is about two orders of magnitude stronger. Indeed, by taking into account that the estimate for the Fermi momenta shift in Eq. (3.27) is proportional to the coupling, we conclude that the chiral asymmetry should be of the order of 1 MeV in dense quark matter. Such a large asymmetry may in turn produce a substantial neutrino emission asymmetry with observable consequences for protoneutron stars [100].

3.4 Discussion

In this chapter, we studied the chiral asymmetry induced by a strong external magnetic field in cold dense QED matter. This study extends the general predictions of Ref. [112] regarding the structure of the chiral asymmetry at Fermi surface. Unlike the weak-field analysis of Ref. [112], however, the present chapter addressed the problem in the general framework that relies on the Landau-level representation. Additionally, the screening effects of dense plasma are taken into account in this study. This is done by utilizing the conventional hard-dense-loop approximation, which is justified in the regime considered.

Among the main results are the numerical functions for the chiral shift and the chiral chemical potential. The dependence of both functions on the longitudinal momentum reveals local peaks at the approximate position of the Fermi surface. This feature is in qualitative agreement with the perturbative weak-field results in Ref. [112], where such functions had logarithmic singularities.

Table 3.1: Data for the chiral asymmetry functions Δ_n , $\mu_{5,n}$, and $(p_3 - p_3^{(0)})/\mu$ at the Fermi surface.

n	Δ_n/μ	$\mu_{5,n}/\mu$	$(p_3 - p_3^{(0)})/\mu$
1	$-1.32 \times 10^{-4} \pm 1.84 \times 10^{-7}$	$-5.56 \times 10^{-5} \pm 3.30 \times 10^{-7}$	$\pm 1.90 \times 10^{-4}$
2	$-1.34 \times 10^{-4} \pm 2.56 \times 10^{-7}$	$-5.57 \times 10^{-5} \pm 4.12 \times 10^{-7}$	$\pm 1.93 \times 10^{-4}$
3	$-1.37 \times 10^{-4} \pm 3.21 \times 10^{-7}$	$-5.41 \times 10^{-5} \pm 4.64 \times 10^{-7}$	$\pm 1.97 \times 10^{-4}$
4	$-1.39 \times 10^{-4} \pm 3.76 \times 10^{-7}$	$-5.18 \times 10^{-5} \pm 5.01 \times 10^{-7}$	$\pm 2.00 \times 10^{-4}$
5	$-1.43 \times 10^{-4} \pm 4.34 \times 10^{-7}$	$-5.13 \times 10^{-5} \pm 5.27 \times 10^{-7}$	$\pm 2.05 \times 10^{-4}$
6	$-1.46 \times 10^{-4} \pm 4.90 \times 10^{-7}$	$-4.84 \times 10^{-5} \pm 5.43 \times 10^{-7}$	$\pm 2.09 \times 10^{-4}$
7	$-1.48 \times 10^{-4} \pm 5.46 \times 10^{-7}$	$-4.68 \times 10^{-5} \pm 5.52 \times 10^{-7}$	$\pm 2.12 \times 10^{-4}$
8	$-1.50 \times 10^{-4} \pm 5.97 \times 10^{-7}$	$-4.71 \times 10^{-5} \pm 5.52 \times 10^{-7}$	$\pm 2.17 \times 10^{-4}$
9	$-1.53 \times 10^{-4} \pm 6.54 \times 10^{-7}$	$-4.38 \times 10^{-5} \pm 5.42 \times 10^{-7}$	$\pm 2.22 \times 10^{-4}$
10	$-1.57 \times 10^{-4} \pm 7.12 \times 10^{-7}$	$-4.05 \times 10^{-5} \pm 5.23 \times 10^{-7}$	$\pm 2.27 \times 10^{-4}$
11	$-1.58 \times 10^{-4} \pm 7.53 \times 10^{-7}$	$-3.76 \times 10^{-5} \pm 4.94 \times 10^{-7}$	$\pm 2.31 \times 10^{-4}$
12	$-1.60 \times 10^{-4} \pm 8.01 \times 10^{-7}$	$-3.18 \times 10^{-5} \pm 4.48 \times 10^{-7}$	$\pm 2.31 \times 10^{-4}$
13	$-1.63 \times 10^{-4} \pm 8.47 \times 10^{-7}$	$-2.69 \times 10^{-5} \pm 3.83 \times 10^{-7}$	$\pm 2.37 \times 10^{-4}$
14	$-1.66 \times 10^{-4} \pm 8.96 \times 10^{-7}$	$-1.92 \times 10^{-5} \pm 2.84 \times 10^{-7}$	$\pm 2.40 \times 10^{-4}$

The values of the chiral shift Δ_n and the chiral chemical potential $\mu_{5,n}$ at the Fermi surface appear to be of order $\alpha|eB|/\mu$. This differs from the corresponding weak-field prediction $\alpha|eB|\mu/m^2$ by a rather large factor $(\mu/m)^2$. Such a difference is not surprising and, in fact, should have been expected in the ultra relativistic limit when $|eB|/m^2$ is not a good expansion parameter. While the dependence of Δ_n on the Landau-level index n shows a weak growth, $\mu_{5,n}$ decreases with n . By fitting the numerical results, we proposed simple model functions which describe the results quite well.

By making use of our numerical results for Δ_n and $\mu_{5,n}$, we also obtained the interaction induced deviations of the Fermi momenta $(p_3 - p_3^{(0)})/\mu$ for the predominantly left-handed and right-handed fermions. These provide the formal measure of the chiral asymmetry at the Fermi surface. The corresponding values appear to be rather small in the case of dense QED matter even at extremely large densities and extremely strong magnetic fields. We suggest, however, that the results can be substantial in the case of quark matter.

Chapter 4

SUMMARY AND OUTLOOK

4.1 Summary

As emphasized in the introduction, the phenomenon of magnetic catalysis has important implications in explaining graphene quantum Hall effect resulting from the broken flavor symmetry due to long-range Coulomb interaction, and in chiral asymmetry of cold QED plasma in a weak magnetic field.

In chapter 2, we studied the effect of static screening of long-range Coulomb interaction on the dynamical parameters of graphene quantum Hall states at different integer filling factors. It is analyzed in the framework of a low-energy Dirac theory for graphene quasiparticles in the mean-field approximation. The generalized Landau-level representation is used for the quasiparticle propagators and the Schwinger-Dyson (gap) equations. One of the main feature of the model is the running dynamical parameters with the Landau levels index. The numerical results demonstrated that the static screening leads to a substantial suppression of the dynamical parameters in all quantum Hall states with respect to the results ignoring the screening effects Ref. [89]. The largest absolute values of the dynamical parameters are typically located at Landau level near the Fermi energy and dynamical parameters approach the bare values at large LL index n . The analysis of the quasiparticle energies and the symmetry properties showed that, the states with the filling factors $\nu = 4n + 2$ and $\nu = 4n$ have the $U_{\uparrow}(2) \times U_{\downarrow}(2)$ symmetry, characterized by the singlet type of the QHF and MC order parameters with respect to $U_s(2)$ ($s = \downarrow, \uparrow$). The states with the odd filling factors $\nu = 4n \pm 1$ are characterized by symmetries, $U_{\uparrow}(1) \times U_{\uparrow}(1) \times U_{\downarrow}(2)$ or $U_{\uparrow}(2) \times U_{\downarrow}(1) \times U_{\downarrow}(1)$, with both the singlet and triplet order parameters.

The dependence of the energy gaps on temperature in QH states with filling factors

$\nu = 4n$ and $\nu = 4n \pm 1$ was also studied. It was found that the symmetry breaking (triplet) order parameters, describing the $\nu = 4n \pm 1$ states, vanish at a certain critical value of temperature T_c , where a second-order or weakly first-order transition occurs. The $\nu = 4n$ states show a smooth crossover and no symmetry-restoring phase transition in the temperature dependence of the energy gaps, which is consistent with the fact that the corresponding states have no real symmetry-breaking order parameters. The Landau-level running of the quasiparticle dynamical parameters could be probed via a systematic study of experimental optical transition lines such as Refs. [143, 144, 145]. The temperature dependence of the energy gaps obtained in this study appears to be in a qualitative agreement with the measurements of the activation energies in Refs. [70, 71, 123, 83, 84, 128, 129]. While in order to achieve quantitative agreement with the experimental data, one needs to further improve the theoretical model.

In chapter 3, we extended the analytical study of the chiral asymmetry of cold dense QED matter at Fermi surface at weak magnetic field [112] to a strong external magnetic field in the framework of generalized Landau-level representation. The screening effects of dense plasma were taken into account by utilizing the conventional hard-dense-loop approximation of the photon propagator. The chiral shift Δ_n and the parity-even chiral chemical potential function $\mu_{5,n}$ were derived from the fermion self-energy and solved numerically. The numerical results showed that both functions peaks at the approximate position of the Fermi surface. This is in qualitative agreement with the perturbative weak-field results in [112], where such functions had logarithmic singularities. The dependence of Δ_n on the Landau-level index n shows a weak growth and $\mu_{5,n}$ decreases with n . The values of the chiral shift Δ_n and the chiral chemical potential $\mu_{5,n}$ at the Fermi surface appear to be of order $\alpha|eB|/\mu$. This differs from the corresponding weak-field prediction $\alpha|eB|\mu/m^2$ by a rather large factor $(\mu/m)^2$. Such a difference is expected in the ultra relativistic limit when $|eB|/m^2$ is not a good expansion parameter. From the chiral shift

and the chiral chemical potential, the chiral asymmetry, which is the interaction induced deviations of the Fermi momenta ($p_3 - p_3^{(0)}$), for the predominantly left-handed and right-handed fermions were obtained. The corresponding values appear to be rather small in the case of dense QED matter even at extremely large densities and extremely strong magnetic fields. We suggest, however, that the results can be substantial in the case of quark matter.

4.2 Outlook

In the study of the graphene quantum Hall effect, there remains a question about the role of small short-range four-fermion and six-fermion interactions [122], which are subject to a large renormalization and, thus, could be even more important than the Zeeman interaction in breaking flavor symmetry. The competition of various short-range interactions gives rise to different ground states. It was showed [137] in the framework of quantum Hall ferromagnetism that the valley-sublattice asymmetric short-range interactions, though much smaller than the long-range Coulomb interaction, give rise to isospin anisotropy and the signs and values of the short-range coupling constants play crucial role in favoring different phases for the $\nu = 0$ ground state: spin-polarized ferromagnetic (F), charge density wave (CDW), Kekule distortion (KD), and canted antiferromagnetic (CAF), where CDW, KD, and CAF phases have gapped edge excitations and exhibit insulating behavior and F phase has gapless edge excitations.

The phase diagram induced by the contact interactions in constant magnetic and pseudomagnetic fields was studied [174] for the $\nu = 0$ ground state in the low-energy effective theory. Including both the long-range Coulomb interaction, the short-range contact interactions and the static screening effect in the framework of low-energy effective theory introduced in chapter 2, the numerical analysis of the quasiparticle gap equation involves a larger set of order parameters as functions of the Pauli matrices than that in chapter 2. The results will shed lights on the preferred ground state phases and transitions for the quantum

Hall states at any integer filling factors, which is related to the coupling constants and the tilt of the magnetic field.

The chiral asymmetry of opposite chirality fermions obtained in chapter 3 may have important implications in neutron stars with dense relativistic fermions (such as electron, quark plasma) in rather strong magnetic fields. The chiral asymmetry in stellar plasma could provide a mechanism for the observed stellar properties such as the pulsar kicks with large spatial velocities of the order of 1000 km/s. The asymmetric Fermi surface of the left-handed charged fermions is carried away by neutrino diffusion after interaction with thermally equilibrated left-handed fermions of the plasma, as the weak processes involving only left-handed fermions. The chiral shift in hot magnetized plasma could also provide a mechanism of neutrino push triggering supernova explosions, which yet to be worked out.

REFERENCES

- [1] J. Charbonneau, A. Zhitnitsky, JCAP **8**, 010 (2010)
- [2] J. Charbonneau, K. Hoffman, J. Heyl, Mon. Not. Roy. Astron. Soc. **404**(1), 119 (2010)
- [3] A. Ohnishi, N. Yamamoto, arXiv:1402.4760[astro-ph.HE] (2014)
- [4] M. Giovannini, M.E. Shaposhnikov, Phys. Rev. D **57**(4), 2186 (1998)
- [5] A. Boyarsky, J. Fröhlich, O. Ruchayskiy, Phys. Rev. Lett. **108**(3), 031301 (2012)
- [6] H. Tashiro, T. Vachaspati, A. Vilenkin, Phys. Rev. D **86**(10), 105033 (2012)
- [7] M. D’Elia, Lect. Notes Phys. **871**, 181 (2013)
- [8] D.E. Kharzeev, K. Landsteiner, A. Schmitt, H.U. Yee, Lect. Notes Phys. **871**, 1 (2013)
- [9] J. Liao, Pramana **84**(5), 901 (2015)
- [10] B. Abelev, et al., Phys. Rev. Lett. **110**(1), 012301 (2013)
- [11] L. Adamczyk, et al., Phys. Rev. Lett. **113**(5), 052302 (2014)
- [12] E.V. Gorbar, V.P. Gusynin, V.A. Miransky, Low Temp. Phys. **34**(10), 790 (2008)
- [13] E.V. Gorbar, Phys. Rev. B **78**(8), 085437 (2008)
- [14] V.P. Gusynin, V.A. Miransky, S.G. Sharapov, I.A. Shovkovy, Phys. Rev. B **74**(19), 195429 (2006)
- [15] I.F. Herbut, B. Roy, Phys. Rev. B **77**(24), 245438 (2008)
- [16] G.W. Semenoff, F. Zhou, JHEP **7**, 37 (2011)
- [17] G.W. Semenoff, Phys. Rev. B **83**(11), 115450 (2011)
- [18] A.M. Turner, A. Vishwanath, arXiv:1301.0330 (2013)
- [19] O. Vafek, A. Melikyan, M. Franz, Z. Tešanović, Phys. Rev. B **63**(13), 134509 (2001)
- [20] Q. Li, D.E. Kharzeev, C. Zhang, Y. Huang, I. Pletikosic, A.V. Fedorov, R.D. Zhong, J.A. Schneeloch, G.D. Gu, T. Valla, arXiv:1412.6543 (2014)
- [21] V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Rev. Lett. **73**(26), 3499 (1994)

- [22] S.P. Klevansky, R.H. Lemmer, Phys. Rev. D **39**(11), 3478 (1989)
- [23] H. Suganuma, T. Tatsumi, Ann. Phys. **208**(2), 470 (1991)
- [24] V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Rev. D **52**(8), 4718 (1995)
- [25] V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Lett. B **349**(4), 477 (1995)
- [26] V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Rev. D **52**(8), 4747 (1995)
- [27] K.G. Klimenko, Z. Phys. C **54**, 323 (1992)
- [28] K.G. Klimenko, Theor. Math. Phys. **90**, 1 (1992). [Teor. Mat. Fiz.90,3(1992)]
- [29] I.V. Krive, Phys. Rev. D **46**(6), 2737 (1992)
- [30] W. Heisenberg, H. Euler, Z. Phys. **98**(11), 714 (1936)
- [31] J. Schwinger, Phys. Rev. **82**(5), 664 (1951)
- [32] G.V. Dunne, Int. J. Mod. Phys. A **27**(15), 1260004 (2012)
- [33] M. Ishi-i, Phys. Rev. D **65**(6), 065025 (2002)
- [34] P.I. Fomin, V.P. Gusynin, V.A. Miransky, Yu.A. Sitenko, Riv. Nuovo Cim. **6N5**, 1 (1983)
- [35] V.A. Miransky, P.I. Fomin, Sov. J. Part. Nucl. **16**, 469 (1985). [Fiz. Elem. Chast. Atom. Yadra 16, 469 (1985)]
- [36] V.A. Miransky, *Dynamical symmetry breaking in quantum field theories* (Singapore, Singapore: World Scientific, 1994)
- [37] S.P. Klevansky, Rev. Mod. Phys. **64**(3), 649 (1992)
- [38] J. Bardeen, Phys. Rev. **108**(5), 1175 (1957)
- [39] J. Bardeen, Phys. Rev. **106**(1), 162 (1957)
- [40] R.D. Pisarski, Phys. Rev. D **29**(10), 2423 (1984)
- [41] T.W. Appelquist, Phys. Rev. D **33**(12), 3704 (1986)
- [42] A.I. Akhiezer, V.B. Berestetski, *Quantum electrodynamics* (Interscience Publishers, New York, 1965)
- [43] J.O. Andersen, Phys. Rev. D **51**(6), 3073 (1995)
- [44] D. Cangemi, G. Dunne, Ann. Phys. **249**(2), 582 (1996)

- [45] P. Elmfors, Phys. Rev. Lett. **71**(4), 480 (1993)
- [46] P. Elmfors, D. Persson, B.S. Skagerstam, Astropart. Phys. **2**, 299 (1994)
- [47] P. Elmfors, Phys. Rev. D **51**(10), 5885 (1995)
- [48] D. Persson, Phys. Rev. D **51**(4), 2026 (1995)
- [49] N.D. Mermin, H. Wagner, Phys. Rev. Lett. **17**(22), 1133 (1966)
- [50] S.R. Coleman, Commun. Math. Phys. **31**(4), 259 (1973)
- [51] J.M. Kosterlitz, D.J. Thouless, J. Phys. C Solid State Phys. **6**, 1181 (1973)
- [52] V.L. Berezinsky, Sov. Phys. JETP **32**, 493 (1971). [Zh. Eksp. Teor. Fiz.59,907(1971)]
- [53] V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Nucl. Phys. B **563**(1-2), 361 (1999)
- [54] V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Rev. Lett. **83**(7), 1291 (1999)
- [55] V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Nucl. Phys. B **462**(2-3), 249 (1996)
- [56] E.V. Gorbar, V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Rev. B **66**(4), 045108 (2002)
- [57] E.V. Gorbar, Phys. Rev. D **64**(10), 105028 (2001)
- [58] J. Alexandre, Phys. Rev. D **64**(12), 125007 (2001)
- [59] S. Teber, Phys. Rev. D **86**(2), 025005 (2012)
- [60] A.V. Kotikov, Phys. Rev. D **89**(6), 065038 (2014)
- [61] P.R. Wallace, Phys. Rev. **71**(9), 622 (1947)
- [62] L.D. Landau, Phys. Z. Sowjetunion **11**, 26 (1937)
- [63] R.E. Peierls, Ann. I. H. Poincare **5**, 177 (1935)
- [64] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, Y. Zhang, S.V. Dubonos, I.V. Grigorieva, A.A. Firsov, Science **306**, 666 (2004)
- [65] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, M.I. Katsnelson, I.V. Grigorieva, S.V. Dubonos, A.A. Firsov, Nature **438**, 197 (2005)
- [66] A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, A.K. Geim, Rev. Mod. Phys. **81**(1), 109 (2009)
- [67] G.W. Semenoff, Phys. Rev. Lett. **53**(26), 2449 (1984)

- [68] K.S. Novoselov, Z. Jiang, Y. Zhang, S.V. Morozov, H.L. Stormer, U. Zeitler, J.C. Maan, G.S. Boebinger, P. Kim, A.K. Geim, *Science* **315**(5817), 1379 (2007)
- [69] J.W. McClure, *Phys. Rev.* **104**(3), 666 (1956)
- [70] Y. Zhang, *Phys. Rev. Lett.* **96**(13), 136806 (2006)
- [71] Z. Jiang, *Phys. Rev. Lett.* **99**(10), 106802 (2007)
- [72] D.A. Abanin, P.A. Lee, L.S. Levitov, *Phys. Rev. Lett.* **96**(17), 176803 (2006)
- [73] D.A. Abanin, *Phys. Rev. Lett.* **98**(15), 156801 (2007)
- [74] K. Nomura, *Phys. Rev. Lett.* **96**(25), 256602 (2006)
- [75] J. Alicea, *Phys. Rev. B* **74**(7), 075422 (2006)
- [76] K. Yang, *Phys. Rev. B* **74**(7), 075423 (2006)
- [77] M.O. Goerbig, *Phys. Rev. B* **74**(16), 161407 (2006)
- [78] I.F. Herbut, *Phys. Rev. B* **75**(16), 165411 (2007)
- [79] I.F. Herbut, *Phys. Rev. B* **76**(8), 085432 (2007)
- [80] M. Ezawa, *J. Phys. Soc. of Japan* **76**(9), 094701 (2007)
- [81] M. Ezawa, *Physica E* **40**(2), 269 (2007)
- [82] K. Yang, *Solid State Commun.* **143**, 27 (2007)
- [83] X. Du, I. Skachko, F. Duerr, A. Luican, E.Y. Andrei, *Nature* **462**, 192 (2009)
- [84] K.I. Bolotin, F. Ghahari, M.D. Shulman, H.L. Stormer, P. Kim, *Nature* **462**, 196 (2009)
- [85] D.A. Abanin, *Phys. Rev. Lett.* **98**(19), 196806 (2007)
- [86] I.F. Herbut, *Phys. Rev. Lett.* **97**(14), 146401 (2006)
- [87] J.N. Fuchs, *Phys. Rev. Lett.* **98**(1), 016803 (2007)
- [88] V.P. Gusynin, S.G. Sharapov, J.P. Carbotte, *Int. J. Mod. Phys. B* **21**(27), 4611 (2007)
- [89] E.V. Gorbar, V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, *Phys. Script. T* **146**, 014018 (2012)
- [90] A.M. Bernstein, B.R. Holstein, *Rev. Mod. Phys.* **85**(1), 49 (2013)
- [91] J.S. Bell, R. Jackiw, *Nuovo Cim. A* **60**(60), 47 (1969)

- [92] S.L. Adler, Phys. Rev. **177**(5), 2426 (1969)
- [93] G. Veneziano, Nucl. Phys. B **159**(1-2), 213 (1979)
- [94] E. Witten, Nucl. Phys. B **156**(2), 269 (1979)
- [95] A. Vilenkin, Phys. Rev. D **22**(12), 3080 (1980)
- [96] M.A. Metlitski, A.R. Zhitnitsky, Phys. Rev. D **72**(4), 045011 (2005)
- [97] G.M. Newman, D.T. Son, Phys. Rev. D **73**(4), 045006 (2006)
- [98] D.T. Son, Phys. Rev. D **70**(7), 074018 (2004)
- [99] E.V. Gorbar, V.A. Miransky, I.A. Shovkovy, Phys. Rev. C **80**(3), 032801 (2009)
- [100] E.V. Gorbar, V.A. Miransky, I.A. Shovkovy, Phys. Rev. D **83**(8), 085003 (2011)
- [101] E.V. Gorbar, V.A. Miransky, I.A. Shovkovy, X. Wang, Phys. Rev. D **88**(2), 025025 (2013)
- [102] L.D. Kharzeev, D. E. McLerran, H.J. Warringa, Nucl. Phys. A **803**, 227 (2008)
- [103] D. Kharzeev, A. Zhitnitsky, Nucl. Phys. A **797**, 67 (2007)
- [104] K. Fukushima, Phys. Rev. D **78**(7), 074033 (2008)
- [105] D. Kharzeev, Phys. Lett. B **633**, 260 (2006)
- [106] J. Ambjørn, J. Greensite, C. Peterson, Nucl. Phys. B **221**(2), 381 (1983)
- [107] D.E. Kharzeev, H.U. Yee, Phys. Rev. D **83**(8), 085007 (2011)
- [108] E.V. Gorbar, V.A. Miransky, I.A. Shovkovy, Phys. Lett. B **695**(1-4), 354 (2011)
- [109] Y. Burnier, D.E. Kharzeev, J. Liao, H.U. Yee, Phys. Rev. Lett. **107**(5), 052303 (2011)
- [110] K. Fukushima, M. Ruggieri, Phys. Rev. D **82**(5), 054001 (2010)
- [111] E. Gorbar, V. Miransky, I. Shovkovy, Prog. Theor. Phys. Suppl. **186**, 471 (2010)
- [112] E.V. Gorbar, V.A. Miransky, I.A. Shovkovy, X. Wang, Phys. Rev. D **88**(2), 025043 (2013)
- [113] V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Nucl. Phys. B **462**(2-3), 249 (1996)
- [114] T. Kojo, Y. Hidaka, L. McLerran, R.D. Pisarski, Nucl. Phys. A **843**(1-4), 37 (2010)
- [115] D.P. DiVincenzo, Phys. Rev. B **29**(4), 1685 (1984)
- [116] Y. Zhang, Y.W. Tan, H.L. Stormer, P. Kim, Nature **438**, 201 (2005)

- [117] Y. Zheng, Phys. Rev. B **65**(24), 245420 (2002)
- [118] V.P. Gusynin, Phys. Rev. Lett. **95**(14), 146801 (2005)
- [119] N.M.R. Peres, Phys. Rev. B **73**(12), 125411 (2006)
- [120] D.S.L. Abergel, V. Apalkov, J. Berashevich, K. Ziegler, T. Chakraborty, Adv. Phys. **59**(4), 261 (2010)
- [121] Y. Barlas, K. Yang, A.H. MacDonald, Nanotech. **23**(5), 052001 (2012)
- [122] I.L. Aleiner, Phys. Rev. B **76**(19), 195415 (2007)
- [123] J.G. Checkelsky, Phys. Rev. Lett. **100**(20), 206801 (2008)
- [124] A.J.M. Giesbers, Phys. Rev. B **80**(20), 201403 (2009)
- [125] J.G. Checkelsky, Phys. Rev. B **79**(11), 115434 (2009)
- [126] L. Zhang, Y. Zhang, M. Khodas, T. Valla, I.A. Zaliznyak, Phys. Rev. Lett. **105**(4), 046804 (2010)
- [127] Y. Zhao, Phys. Rev. Lett. **108**(10), 106804 (2012)
- [128] A.F. Young, C.R. Dean, L. Wang, H. Ren, P. Cadden-Zimansky, K. Watanabe, T. Taniguchi, J. Hone, K.L. Shepard, P. Kim, Nature Phys. **8**, 550 (2012)
- [129] A.F. Young, J.D. Sanchez-Yamagishi, B. Hunt, S.H. Choi, K. Watanabe, T. Taniguchi, R.C. Ashoori, P. Jarillo-Herrero, Nature **505**, 528 (2014)
- [130] F. Chiappini, Phys. Rev. B **92**(20), 201412 (2015)
- [131] L. Sheng, Phys. Rev. Lett. **99**(19), 196802 (2007)
- [132] M.M. Fogler, Phys. Rev. B **52**(24), 17366 (1995)
- [133] I.A. Shovkovy, Lect. Notes Phys. **871**, 13 (2013)
- [134] V.A. Miransky, I.A. Shovkovy, Phys. Rep **576**, 1 (2015)
- [135] D.V. Khveshchenko, Phys. Rev. Lett. **87**(20), 206401 (2001)
- [136] I.F. Herbut, Phys. Rev. B **79**(8), 085116 (2009)
- [137] M. Kharitonov, Phys. Rev. B **85**(15), 155439 (2012)
- [138] J. Jung, Phys. Rev. B **80**(23), 235417 (2009)
- [139] K. Nomura, Phys. Rev. Lett. **103**(21), 216801 (2009)

- [140] B. Roy, Phys. Rev. B **90**(20), 201409 (2014)
- [141] P.K. Pyatkovskiy, V.P. Gusynin, Phys. Rev. B **83**(7), 075422 (2011)
- [142] E.V. Gorbar, V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Rev. B **85**(23), 235460 (2012)
- [143] M.L. Sadowski, Phys. Rev. Lett. **97**(26), 266405 (2006)
- [144] Z. Jiang, Phys. Rev. Lett. **98**(19), 197403 (2007)
- [145] M. Orlita, M. Potemski, Semicond. Science Tech. **25**, 063001 (2010)
- [146] A. Iyengar, Phys. Rev. B **75**(12), 125430 (2007)
- [147] D.E. Kharzeev, Prog. Part. Nucl. Phys. **75**, 133 (2014)
- [148] O. Vafek, A. Vishwanath, Ann. Rev. Cond. Mat. Phys. **5**, 83 (2013)
- [149] D.F. Hou, H. Liu, H.C. Ren, Phys. Rev. D **86**(12), 121703 (2012)
- [150] K. Fukushima, Lect. Notes Phys. **871**, 241 (2013)
- [151] A. Jimenez-Alba, K. Landsteiner, L. Melgar, Phys. Rev. D **90**(12), 126004 (2014)
- [152] L. Adamczyk, J.K. Adkins, G. Agakishiev, M.M. Aggarwal, Z. Ahammed, A.V. Alakhverdyants, I. Alekseev, J. Alford, C.D. Anson, D. Arkhipkin, et al., Phys. Rev. C **89**(4), 044908 (2014)
- [153] G. Wang, Nucl. Phys. A **904**, 248 (2013)
- [154] H. Ke, Star Collaboration, J. Phys. Conf. Series **389**, 012035 (2012)
- [155] J. Erdmenger, M. Haack, M. Kaminski, A. Yarom, JHEP **1**, 055 (2009)
- [156] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, P. Surówka, JHEP **1**, 94 (2011)
- [157] D.T. Son, P. Surówka, Phys. Rev. Lett. **103**(19), 191601 (2009)
- [158] Y. Neiman, Y. Oz, JHEP **09**, 011 (2011)
- [159] S. Bhattacharyya, S. Jain, S. Minwalla, T. Sharma, JHEP **1**, 40 (2013)
- [160] A. Jimenez-Alba, L. Melgar, JHEP **10**, 120 (2014)
- [161] E.V. Gorbar, V.A. Miransky, I.A. Shovkovy, Phys. Rev. B **88**(16), 165105 (2013)
- [162] S. Golkar, D.T. Son, JHEP **2**, 169 (2015)

- [163] K. Jensen, P. Kovtun, A. Ritz, JHEP **10**, 186 (2013)
- [164] V.P. Kirilin, A.V. Sadofyev, V.I. Zakharov, arXiv:1312.0895 (2013)
- [165] P. Watson, H. Reinhardt, Phys. Rev. D **89**(4), 045008 (2014)
- [166] M.E. Peskin, D.V. Schroeder, *An Introduction to quantum field theory* (Westview Press, Boulder, Colorado, 1995)
- [167] H. Vija, M.H. Thoma, Phys. Lett. B **342**(1-4), 212 (1995)
- [168] C. Manuel, Phys. Rev. D **53**(10), 5866 (1996)
- [169] A. Chodos, Phys. Rev. D **42**(8), 2881 (1990)
- [170] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, Orlando, 1994)
- [171] B.A. Freedman, L.D. McLerran, Phys. Rev. D **16**(4), 1130 (1977)
- [172] S. Weinzierl, arXiv:hep-ph/0006269 (2000)
- [173] Richard chandler's source code for generating gamma-distributed random numbers.
URL <http://www.ucl.ac.uk/~ucakarc/work/software/randgen.f>
- [174] D.O. Rybalka, E.V. Gorbar, V.P. Gusynin, Phys. Rev. B **91**(11), 115132 (2015)

APPENDIX A

FERMION PROPAGATOR WITH CHIRAL ASYMMETRY

The fermion propagator with the nonzero Dirac mass, chiral shift, and chiral chemical potential is formally defined by

$$G(u, u') = i\langle u | \left[(i\partial_t + \mu)\gamma^0 + \Delta\gamma^3\gamma^5 + \mu_5\gamma^0\gamma^5 - (\pi_\perp \cdot \gamma) - \pi^3\gamma^3 - m \right]^{-1} | u' \rangle, \quad (1.1)$$

where $u = (t, \mathbf{r})$ is a space-time four-vector. Applying simple calculation on the fermion propagator, we have

$$\begin{aligned} G(u, u') &= i\langle u | \left[(i\partial_t + \mu)\gamma^0 + \Delta\gamma^3\gamma^5 + \mu_5\gamma^0\gamma^5 - (\pi_\perp \cdot \gamma) - \pi^3\gamma^3 - m \right]^{-1} | u' \rangle \\ &= i\langle u | \left[(i\partial_t + \mu)\gamma^0 - \Delta\gamma^3\gamma^5 - \mu_5\gamma^0\gamma^5 - (\pi_\perp \cdot \gamma) - \pi^3\gamma^3 + m \right] \\ &\quad \times \left\{ \left[(i\partial_t + \mu)\gamma^0 + \Delta\gamma^3\gamma^5 + \mu_5\gamma^0\gamma^5 - (\pi_\perp \cdot \gamma) - \pi^3\gamma^3 - m \right] \right. \\ &\quad \times \left. \left[(i\partial_t + \mu)\gamma^0 - \Delta\gamma^3\gamma^5 - \mu_5\gamma^0\gamma^5 - (\pi_\perp \cdot \gamma) - \pi^3\gamma^3 + m \right] \right\}^{-1} | u' \rangle \\ &= i\langle u | \left[(i\partial_t + \mu)\gamma^0 - \Delta\gamma^3\gamma^5 - \mu_5\gamma^0\gamma^5 - (\pi_\perp \cdot \gamma) - \pi^3\gamma^3 + m \right] \\ &\quad \times \left[(i\partial_t + \mu)^2 - \Delta^2 + \mu_5^2 - \pi_\perp^2 - \pi_3^2 - m^2 - 2[\Delta k_3 + (i\partial_t + \mu)\mu_5]\gamma^5 \right. \\ &\quad \left. + 2m\mu_5\gamma^0\gamma^5 + 2m\Delta\gamma^3\gamma^5 - ieB\gamma^1\gamma^2 \right]^{-1} | u' \rangle \end{aligned} \quad (1.2)$$

In the Fourier transformation in time t and in the z -direction (along the magnetic field), the full fermion propagator reads

$$\begin{aligned} G(\omega, k^3; \mathbf{r}, \mathbf{r}') &= \int dt dz e^{i\omega(t-t') - ik^3(z-z')} G(u, u') \\ &= i \left[W - (\pi_\perp \cdot \gamma) \right] \langle \mathbf{r} | (\mathcal{M} - \pi_\perp^2 - ieB\gamma^1\gamma^2)^{-1} | \mathbf{r}' \rangle \end{aligned} \quad (1.3)$$

where

$$\begin{aligned} W &= (\omega + \mu)\gamma^0 - \Delta\gamma^3\gamma^5 - \mu_5\gamma^0\gamma^5 - k^3\gamma^3 + m \\ \mathcal{M} &= (\omega + \mu)^2 - \Delta^2 + \mu_5^2 - k_3^2 - m^2 \\ &\quad - 2[\Delta k_3 + (\omega + \mu)\mu_5]\gamma^5 + 2m\Delta\gamma^3\gamma^5 + 2m\mu_5\gamma^0\gamma^5. \end{aligned} \quad (1.4)$$

Under the common basis of the three mutually commute operators γ^0 , $\pi_\perp^2$ and $ieB\gamma^1\gamma^2$, the eigenvalues of $\pi_\perp^2$ are $2(n+1)|eB|$, $n = 0, 1, 2, \dots$

$$\begin{aligned}\langle \mathbf{r} | \mathcal{M} - \pi_\perp^2 - ieB\gamma^1\gamma^2)^{-1} | \mathbf{r}' \rangle &= \frac{e^{i\Phi(\mathbf{r}, \mathbf{r}')}}{2\pi l^2} e^{-\xi/2} \sum_{n=0}^{\infty} \frac{L_n(\xi)}{\mathcal{M} - (2n+1)|eB| - ieB\gamma^1\gamma^2} \\ &= \frac{e^{i\Phi(\mathbf{r}, \mathbf{r}')}}{2\pi l^2} e^{-\xi/2} \sum_{n=0}^{\infty} \frac{\mathcal{P}_{-L_n}(\xi) - \mathcal{P}_{+L_{n-1}}(\xi)}{\mathcal{M} - 2n|eB|} \quad (1.5)\end{aligned}$$

where $\Phi(\mathbf{r}, \mathbf{r}')$ is the Schwinger phase and $\xi \equiv (\mathbf{r} - \mathbf{r}')^2 / (2l^2)$.

By making use of this definition and utilizing the same method as in Ref. [100], we straightforwardly show that the fermion propagator has the general structure as in Eq. (3.2) and the explicit form of the translation invariant part is determined by the following expression of its Fourier transform:

$$\begin{aligned}\bar{G}(\omega, \mathbf{k}) &= \int dt d^3\mathbf{r} e^{i\omega t - i(\mathbf{k} \cdot \mathbf{r})} \bar{G}(t; \mathbf{r}) \\ &= ie^{-k_\perp^2 l^2} \sum_{n=0}^{\infty} (-1)^n D_n(\omega, \mathbf{k}) \frac{1}{\mathcal{M} - 2n|eB|}, \quad (1.6)\end{aligned}$$

where $\mathbf{k}_\perp = (k^1, k^2)$, and the n th Landau level contribution is given in terms of the following matrix functions:

$$\begin{aligned}D_n(\omega, \mathbf{k}) &= 2W \left[\mathcal{P}_{-L_n}(2k_\perp^2 l^2) - \mathcal{P}_{+L_{n-1}}(2k_\perp^2 l^2) \right] \\ &\quad + 4(\mathbf{k}_\perp \cdot \boldsymbol{\gamma}) L_{n-1}^1(2k_\perp^2 l^2), \quad (1.7)\end{aligned}$$

Note that the last matrix factor in Eq. (1.6) can be rewritten in the following equivalent form:

$$\begin{aligned}&\frac{1}{\mathcal{M} - 2n|eB|} \quad (1.8) \\ &= \frac{1}{[(\omega + \mu)^2 - \Delta^2 + \mu_5^2 - k_3^2 - m^2 - 2n|eB|] - 2[\Delta k_3 + (\omega + \mu)\mu_5]\gamma^5 + 2m\mu_5\gamma^0\gamma^5 + 2m\Delta\gamma^3\gamma^5} \\ &= \frac{[(\omega + \mu)^2 - \Delta^2 + \mu_5^2 - k_3^2 - m^2 - 2n|eB|] + 2[\Delta k_3 + (\omega + \mu)\mu_5]\gamma^5 - 2m\mu_5\gamma^0\gamma^5 - 2m\Delta\gamma^3\gamma^5}{[(\omega + \mu)^2 - \Delta^2 + \mu_5^2 - k_3^2 - m^2 - 2n|eB|]^2 - 4[\Delta k_3 + (\omega + \mu)\mu_5]^2 + 4m^2\mu_5^2 - 4m^2\Delta^2},\end{aligned}$$

which implies that the fermion dispersion relation in the presence of the chiral asymmetry is determined by the solutions to the equation:

$$\left[(\omega + \mu)^2 - \Delta^2 + \mu_5^2 - k_3^2 - m^2 - 2n|eB| \right]^2 - 4[\Delta k_3 + (\omega + \mu)\mu_5]^2 + 4m^2\mu_5^2 - 4m^2\Delta^2 = 0. \quad (1.9)$$

APPENDIX B

CHIRAL ASYMMETRY FUNCTIONS IN THE HDL APPROXIMATION

In this appendix, we present the explicit form of the chiral asymmetry functions $\Delta_n(p_3)$ and $\mu_{5,n}(p_3)$ in the approximation with the HDL photon propagator. The chiral asymmetry functions Δ_n and $\mu_{5,n}$ are given by

$$\begin{aligned}\Delta_n &= \frac{(-1)^n l^2}{8\pi} \text{sign}(eB) \int d^2 p_\perp e^{-p_\perp^2 l^2} \left[L_n(2p_\perp^2 l^2) + L_{n-1}(2p_\perp^2 l^2) \right] \text{Tr} \left[\gamma^0 \bar{\Sigma}(p) \right] \\ &\quad - \frac{(-1)^n l^2}{8\pi} \int d^2 p_\perp e^{-p_\perp^2 l^2} \left[L_n(2p_\perp^2 l^2) - L_{n-1}(2p_\perp^2 l^2) \right] \text{Tr} \left[\gamma^3 \gamma^5 \bar{\Sigma}(p) \right],\end{aligned}\quad (2.10)$$

$$\begin{aligned}\mu_{5,n} &= \frac{(-1)^n l^2}{8\pi} \text{sign}(eB) \int d^2 p_\perp e^{-p_\perp^2 l^2} \left[L_n(2p_\perp^2 l^2) + L_{n-1}(2p_\perp^2 l^2) \right] \text{Tr} \left[\gamma^3 \bar{\Sigma}(p) \right] \\ &\quad + \frac{(-1)^n l^2}{8\pi} \int d^2 p_\perp e^{-p_\perp^2 l^2} \left[L_n(2p_\perp^2 l^2) - L_{n-1}(2p_\perp^2 l^2) \right] \text{Tr} \left[\gamma^0 \gamma^5 \bar{\Sigma}(p) \right],\end{aligned}\quad (2.11)$$

where the self-energy takes the form

$$\bar{\Sigma}(p) = -4i\pi\alpha \int \frac{dk_0 dk_3 d^2 \mathbf{k}_\perp}{(2\pi)^4} \gamma^\mu \bar{S}(k) \gamma^\nu D_{\mu\nu}(k-p), \quad (2.12)$$

and the HDL propagator in the Coulomb gauge reads

$$D_{\mu\nu}(q) \simeq i \left(\frac{|\mathbf{q}|}{|\mathbf{q}|^3 + \frac{\pi}{4} m_D^2 |q_4|} O_{\mu\nu}^{(\text{mag})} + \frac{O_{\mu\nu}^{(\text{el})}}{q_4^2 + |\mathbf{q}|^2 + m_D^2} \right), \quad (2.13)$$

where $q_4 \equiv iq_0$ and $m_D^2 = 2\alpha\mu_0^2/\pi$ is the Debye screening mass in two-flavor quark matter.

The transverse Lorentz projector $P_{\mu\nu}^T$ is defined as follows:

$$O_{\mu\nu}^{(\text{mag})} = g_{\mu\nu} - u_\mu u_\nu + \frac{q_\mu q_\nu}{|\mathbf{q}|^2}, \quad (2.14)$$

$$O_{\mu\nu}^{(\text{el})} = u_\mu u_\nu. \quad (2.15)$$

where $u_\mu = (1, 0, 0, 0)$. Plugging the translation invariant part of the fermion propagator

and and the HDL photon propagator Eq. 4.2, the self-energy has the form

$$\begin{aligned}
& \bar{\Sigma}(p) \\
&= -4i\pi\alpha \int \frac{dk_0 dk_3 d^2\mathbf{k}_\perp}{(2\pi)^4} \gamma^\mu \bar{S}(k) \gamma^\nu D_{\mu\nu}(k-p) \\
&= 4\pi\alpha \int \frac{dk_0 dk_3 d^2\mathbf{k}_\perp}{(2\pi)^4} \gamma^\mu \bar{S}(k) \gamma^\nu \\
&\quad \times \left(\frac{g_{\mu\nu} - u_\mu u_\nu + \frac{(\mathbf{k}-\mathbf{p})_\mu (\mathbf{k}-\mathbf{p})_\nu}{|\mathbf{k}-\mathbf{p}|^2}}{|\mathbf{k}-\mathbf{p}|^2 + \frac{\pi}{4} m_D^2 \frac{|ik_0 - ip_0|}{|\mathbf{k}-\mathbf{p}|}} + \frac{u_\mu u_\nu}{(ik_0 - ip_0)^2 + |\mathbf{k}-\mathbf{p}|^2 + m_D^2} \right) \\
&= 8\pi\alpha i \int \frac{dk_0 dk_3 d^2\mathbf{k}_\perp}{(2\pi)^4} e^{-k_\perp^2 l^2} \sum_{n=0}^{\infty} (-1)^n \\
&\quad \times \gamma^\mu \left(\frac{[(k_0 + \mu)\gamma^0 - k_3\gamma^3 + m] [\mathcal{P}_- L_n(2k_\perp^2 l^2) - \mathcal{P}_+ L_{n-1}(2k_\perp^2 l^2)]}{[k_0 + \mu + i\varepsilon \text{sgn}(k_0)]^2 - k_3^2 - 2n|eB| - m^2} \right. \\
&\quad \left. + \frac{2(\mathbf{k}_\perp \cdot \boldsymbol{\gamma}_\perp) L_{n-1}^1(2k_\perp^2 l^2)}{[k_0 + \mu + i\varepsilon \text{sgn}(k_0)]^2 - k_3^2 - 2n|eB| - m^2} \right) \gamma^\nu \\
&\quad \times \left(\frac{g_{\mu\nu} - u_\mu u_\nu + \frac{(\mathbf{k}-\mathbf{p})_\mu (\mathbf{k}-\mathbf{p})_\nu}{|\mathbf{k}-\mathbf{p}|^2}}{|\mathbf{k}-\mathbf{p}|^2 + \frac{\pi}{4} m_D^2 \frac{|ik_0 - ip_0|}{|\mathbf{k}-\mathbf{p}|}} + \frac{u_\mu u_\nu}{(ik_0 - ip_0)^2 + |\mathbf{k}-\mathbf{p}|^2 + m_D^2} \right) \quad (2.16)
\end{aligned}$$

Analyzing the Dirac structure of the integrand and keep only the terms of γ^0 , $\gamma^3\gamma^5$, γ^3 , $\gamma^0\gamma^5$,

which contribute to the chiral asymmetry functions $\Delta_n(p_3)$ and $\mu_{5,n}(p_3)$, we have

$$\begin{aligned}
& \gamma^\mu \left(\left[(k_0 + \mu) \gamma^0 - k_3 \gamma^3 + m \right] \left[\mathcal{P}_- L_n(2k_\perp^2 l^2) - \mathcal{P}_+ L_{n-1}(2k_\perp^2 l^2) \right] + 2(\mathbf{k}_\perp \cdot \boldsymbol{\gamma}_\perp) L_{n-1}^1(2k_\perp^2 l^2) \right) \gamma^\nu \\
& \times \left[A^{(mag)} \left(g_{\mu\nu} - u_\mu u_\nu + \frac{\mathbf{q}_\mu \mathbf{q}_\nu}{|\mathbf{q}|^2} \right) + A^{(el)} u_\mu u_\nu \right] \\
= & \gamma^\mu \left\{ \frac{1}{2} (k_0 + \mu) \left[L_n(2k_\perp^2 l^2) - L_{n-1}(2k_\perp^2 l^2) \right] \gamma^0 - \frac{\text{sign}(eB)}{2} (k_0 + \mu) \left[L_n(2k_\perp^2 l^2) + L_{n-1}(2k_\perp^2 l^2) \right] \gamma^3 \gamma^5 \right. \\
& - \frac{1}{2} k_3 \left[L_n(2k_\perp^2 l^2) - L_{n-1}(2k_\perp^2 l^2) \right] \gamma^3 + \frac{\text{sign}(eB)}{2} k_3 \left[L_n(2k_\perp^2 l^2) + L_{n-1}(2k_\perp^2 l^2) \right] \gamma^0 \gamma^5 \\
& + \frac{1}{2} m \left[L_n(2k_\perp^2 l^2) - L_{n-1}(2k_\perp^2 l^2) \right] - \frac{\text{sign}(eB)}{2} m \left[L_n(2k_\perp^2 l^2) + L_{n-1}(2k_\perp^2 l^2) \right] i \gamma^1 \gamma^2 \\
& \left. + 2(\mathbf{k}_\perp \cdot \boldsymbol{\gamma}_\perp) L_{n-1}^1(2k_\perp^2 l^2) \right\} \gamma^\nu \left[A^{(mag)} \left(g_{\mu\nu} - u_\mu u_\nu + \frac{\mathbf{q}_\mu \mathbf{q}_\nu}{|\mathbf{q}|^2} \right) + A^{(el)} u_\mu u_\nu \right] \\
= & \left\{ \left(-A^{(mag)} + \frac{1}{2} A^{(el)} \right) (k_0 + \mu) \left[L_n(2k_\perp^2 l^2) - L_{n-1}(2k_\perp^2 l^2) \right] \gamma^0 \right. \\
& - \left(\frac{q_3^2}{|\mathbf{q}|^2} A^{(mag)} + \frac{1}{2} A^{(el)} \right) (k_0 + \mu) \text{sign}(eB) \left[L_n(2k_\perp^2 l^2) + L_{n-1}(2k_\perp^2 l^2) \right] \gamma^3 \gamma^5 \\
& + \left[\left(\frac{q_3^2}{|\mathbf{q}|^2} A^{(mag)} + \frac{1}{2} A^{(el)} \right) k_3 \left[L_n(2k_\perp^2 l^2) - L_{n-1}(2k_\perp^2 l^2) \right] + 4A^{(mag)} \frac{(k_1 q_1 q_3 + k_2 q_2 q_3)}{|\mathbf{q}|^2} L_{n-1}^1(2k_\perp^2 l^2) \right] \gamma^3 \\
& + \left(-A^{(mag)} + \frac{1}{2} A^{(el)} \right) k_3 \text{sign}(eB) \left[L_n(2k_\perp^2 l^2) + L_{n-1}(2k_\perp^2 l^2) \right] \gamma^0 \gamma^5 \\
& + \left(A^{(mag)} + \frac{1}{2} A^{(el)} \right) m \left[L_n(2k_\perp^2 l^2) - L_{n-1}(2k_\perp^2 l^2) \right] \\
& \left. - \left(-\frac{q_3^2}{|\mathbf{q}|^2} A^{(mag)} + \frac{1}{2} A^{(el)} \right) m \text{sign}(eB) \left[L_n(2k_\perp^2 l^2) + L_{n-1}(2k_\perp^2 l^2) \right] i \gamma^1 \gamma^2 + \dots \right\}, \tag{2.17}
\end{aligned}$$

where $A^{(mag)}$ and $A^{(el)}$ are the factors coming from the multiplications of the denominators and take the form

$$\begin{aligned}
A^{(mag)} &= \frac{1}{[k_0 + \mu + i\epsilon \text{sgn}(k_0)]^2 - k_3^2 - 2n|eB| - m^2} \cdot \frac{1}{|\mathbf{k} - \mathbf{p}|^2 + \frac{\pi}{4} m_D^2 \frac{|ik_0 - ip_0|}{|\mathbf{k} - \mathbf{p}|}} \\
A^{(el)} &= \frac{1}{[k_0 + \mu + i\epsilon \text{sgn}(k_0)]^2 - k_3^2 - 2n|eB| - m^2} \cdot \frac{1}{(ik_0 - ip_0)^2 + |\mathbf{k} - \mathbf{p}|^2 + m_D^2}. \tag{2.18}
\end{aligned}$$

Hence we have the following results for the two coefficient functions of interest:

$$\begin{aligned}
\Delta_n(p_3) &= (-1)^n e^2 l^2 \text{sign}(eB) \int \frac{dk_3 d^2 \mathbf{k}_\perp d^2 \mathbf{p}_\perp}{(2\pi)^4} e^{-k_\perp^2 l^2 - p_\perp^2 l^2} \sum_{N=0}^{\infty} (-1)^N \\
&\times \left\{ \left[L_n(2p_\perp^2 l^2) + L_{n-1}(2p_\perp^2 l^2) \right] \left[L_N(2k_\perp^2 l^2) - L_{N-1}(2k_\perp^2 l^2) \right] \right. \\
&\times \left(-\mathcal{D}^{(mag)} + \frac{1}{2} \mathcal{D}^{(el)} \right) \\
&+ \left[L_n(2p_\perp^2 l^2) - L_{n-1}(2p_\perp^2 l^2) \right] \left[L_N(2k_\perp^2 l^2) + L_{N-1}(2k_\perp^2 l^2) \right] \\
&\times \left. \left(\frac{q_3^2}{|\mathbf{q}|^2} \mathcal{D}^{(mag)} + \frac{1}{2} \mathcal{D}^{(el)} \right) \right\}, \tag{2.19}
\end{aligned}$$

$$\begin{aligned}
\mu_{5,n}(p_3) &= (-1)^n e^2 l^2 \text{sign}(eB) \int \frac{dk_3 d^2 \mathbf{k}_\perp d^2 \mathbf{p}_\perp}{(2\pi)^4} e^{-k_\perp^2 l^2 - p_\perp^2 l^2} \sum_{N=0}^{\infty} (-1)^N \\
&\times \left\{ - \left[L_n(2p_\perp^2 l^2) + L_{n-1}(2p_\perp^2 l^2) \right] \left[L_N(2k_\perp^2 l^2) - L_{N-1}(2k_\perp^2 l^2) \right] \right. \\
&\times k_3 \left(\frac{q_3^2}{|\mathbf{q}|^2} \mathcal{F}^{(mag)} + \frac{1}{2} \mathcal{F}^{(el)} \right) \\
&+ \left[L_n(2p_\perp^2 l^2) - L_{n-1}(2p_\perp^2 l^2) \right] \left[L_N(2k_\perp^2 l^2) + L_{N-1}(2k_\perp^2 l^2) \right] \\
&\times k_3 \left(\mathcal{F}^{(mag)} - \frac{1}{2} \mathcal{F}^{(el)} \right) \\
&+ 4L_{N-1}^1(2k_\perp^2 l^2) \left[L_n(2p_\perp^2 l^2) + L_{n-1}(2p_\perp^2 l^2) \right] \\
&\times \left. \frac{q_3(k_1 q_1 + k_2 q_2)}{|\mathbf{q}|^2} \mathcal{F}^{(mag)} \right\}, \tag{2.20}
\end{aligned}$$

where the explicit expressions for the functions $\mathcal{D}^{(mag)}$, $\mathcal{D}^{(el)}$, $\mathcal{F}^{(mag)}$, and $\mathcal{F}^{(el)}$ are ob-

tained after the integrations over k_0 performed, i.e.,

$$\begin{aligned}
\mathcal{D}^{(mag)} &= \frac{i}{\pi} |\mathbf{q}| \int_{-\infty}^{\infty} \frac{(\omega_E - i\mu) d\omega_E}{[(\omega_E - i\mu)^2 + \mathcal{M}_N^2] (|\mathbf{q}|^3 + \frac{\pi}{4} m_D^2 |\omega_E + ip_0|)} \\
&= \frac{|\mathbf{q}|^4 \text{sign}(\mu) \text{sign}(\mathcal{M}_N^2 - \mu^2)}{2 \left[|\mathbf{q}|^6 + \left(\frac{\pi}{4} m_D^2\right)^2 (\mathcal{M}_N - |\mu|)^2 \right]} \\
&\quad - |\mathbf{q}| \text{sign}(\mu) \frac{\frac{1}{4} m_D^2 (\mathcal{M}_N - |\mu|) \ln \frac{|\mathbf{q}|^3}{\frac{\pi}{4} m_D^2 (\mathcal{M}_N - |\mu|)}}{|\mathbf{q}|^6 + \left(\frac{\pi}{4} m_D^2\right)^2 (\mathcal{M}_N - |\mu|)^2} \\
&\quad - \frac{|\mathbf{q}|^4 \text{sign}(\mu)}{2 \left[|\mathbf{q}|^6 + \left(\frac{\pi}{4} m_D^2\right)^2 (\mathcal{M}_N + |\mu|)^2 \right]} \\
&\quad + |\mathbf{q}| \text{sign}(\mu) \frac{\frac{1}{4} m_D^2 (\mathcal{M}_N + |\mu|) \ln \frac{|\mathbf{q}|^3}{\frac{\pi}{4} m_D^2 (\mathcal{M}_N + |\mu|)}}{|\mathbf{q}|^6 + \left(\frac{\pi}{4} m_D^2\right)^2 (\mathcal{M}_N + |\mu|)^2}, \tag{2.21}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(el)} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(i\omega_E + \mu) d\omega_E}{[(\omega_E - i\mu)^2 + \mathcal{M}_N^2] [(\omega_E - ip_0)^2 + |\mathbf{q}|^2 + m_D^2]} \\
&= \frac{\mu \Theta[\mathcal{M}_N^2 - \mu^2]}{\sqrt{|\mathbf{q}|^2 + m_D^2} \left[\left(\sqrt{|\mathbf{q}|^2 + m_D^2} + \mathcal{M}_N \right)^2 - \mu^2 \right]} \\
&\quad - \frac{\text{sign}(\mu) \Theta \left[\mu^2 - \mathcal{M}_N^2 \right] \left(\sqrt{|\mathbf{q}|^2 + m_D^2} + |\mu| \right)}{\sqrt{|\mathbf{q}|^2 + m_D^2} \left[\left(\sqrt{|\mathbf{q}|^2 + m_D^2} + |\mu| \right)^2 - \mathcal{M}_N^2 \right]}, \tag{2.22}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}^{(mag)} &= \frac{1}{\pi} |\mathbf{q}| \int_{-\infty}^{\infty} \frac{d\omega_E}{[(\omega_E - i\mu)^2 + \mathcal{M}_N^2] (|\mathbf{q}|^3 + \frac{\pi}{4} m_D^2 |\omega_E + ip_0|)} \\
&= \frac{1}{\mathcal{M}_N} \left(\frac{|\mathbf{q}|^4 \text{sign}(\mathcal{M}_N^2 - \mu^2)}{2 \left[|\mathbf{q}|^6 + (\frac{\pi}{4} m_D^2)^2 (\mathcal{M}_N - |\mu|)^2 \right]} \right. \\
&\quad - |\mathbf{q}| \frac{\frac{1}{4} m_D^2 (\mathcal{M}_N - |\mu|) \ln \frac{|\mathbf{q}|^3}{\frac{\pi}{4} m_D^2 |\mathcal{M}_N - |\mu||}}{|\mathbf{q}|^6 + (\frac{\pi}{4} m_D^2)^2 (\mathcal{M}_N - |\mu|)^2} \\
&\quad + \frac{|\mathbf{q}|^4}{2 \left[|\mathbf{q}|^6 + (\frac{\pi}{4} m_D^2)^2 (\mathcal{M}_N + |\mu|)^2 \right]} \\
&\quad \left. - |\mathbf{q}| \frac{\frac{1}{4} m_D^2 (\mathcal{M}_N + |\mu|) \ln \frac{|\mathbf{q}|^3}{\frac{\pi}{4} m_D^2 (\mathcal{M}_N + |\mu|)}}{|\mathbf{q}|^6 + (\frac{\pi}{4} m_D^2)^2 (\mathcal{M}_N + |\mu|)^2} \right), \tag{2.23}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}^{(el)} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega_E}{[(\omega_E - i\mu)^2 + \mathcal{M}_N^2] [(\omega_E - ip_0)^2 + |\mathbf{q}|^2 + m_D^2]} \\
&= - \frac{\Theta[\mu^2 - \mathcal{M}_N^2]}{\sqrt{|\mathbf{q}|^2 + m_D^2} \left[\left(\sqrt{|\mathbf{q}|^2 + m_D^2} + |\mu| \right)^2 - \mathcal{M}_N^2 \right]} \\
&\quad + \frac{\Theta[\mathcal{M}_N^2 - \mu^2] \left(\sqrt{|\mathbf{q}|^2 + m_D^2} + \mathcal{M}_N \right)}{\sqrt{|\mathbf{q}|^2 + m_D^2} \mathcal{M}_N \left[\left(\sqrt{|\mathbf{q}|^2 + m_D^2} + \mathcal{M}_N \right)^2 - \mu^2 \right]}, \tag{2.24}
\end{aligned}$$

where $\mathcal{M}_N^2 = k_3^2 + 2N|eB| + m^2$ and $|\mathbf{q}|^2 = |k_3 - p_3|^2 + k_\perp^2 + p_\perp^2 - 2k_\perp p_\perp \cos \phi$.

While performing the numerical analysis, it is convenient to render the above expressions in a dimensionless form. Therefore, we introduce the following dimensionless functions: $\bar{\Delta}_n \equiv \Delta_n/\mu$ and $\bar{\mu}_{5,n} \equiv \mu_{5,n}/\mu$, as well as the following dimensionless variables:

$x = p_{\perp}/\mu$, $y \equiv k_{\perp}/\mu$, $x_3 \equiv p_3/\mu$, and $y_3 \equiv k_3/\mu$. By using this new notation, we have

$$\begin{aligned}
\bar{\Delta}_n &= (-1)^n \frac{e^2}{b} \text{sign}(eB) \int \frac{dy_3 dy dx d\phi}{(2\pi)^3} e^{-(x^2+y^2)/b} \sum_{N=0}^{\infty} (-1)^N xy \\
&\times \left\{ \left[L_n(2x^2/b) + L_{n-1}(2x^2/b) \right] \left[L_N(2y^2/b) - L_{N-1}(2y^2/b) \right] \right. \\
&\times \left(-\bar{\mathcal{D}}^{(mag)} + \frac{1}{2} \bar{\mathcal{D}}^{(el)} \right) \\
&- \left[L_n(2x^2/b) - L_{n-1}(2x^2/b) \right] \left[L_N(2y^2/b) + L_{N-1}(2y^2/b) \right] \\
&\times \left. \left(\frac{(x_3 - y_3)^2 \bar{\mathcal{D}}^{(mag)}}{(x_3 - y_3)^2 + x^2 + y^2 - 2xy \cos \phi} + \frac{1}{2} \bar{\mathcal{D}}^{(el)} \right) \right\}, \tag{2.25}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mu}_{5,n} &= (-1)^n \frac{e^2}{b} \text{sign}(eB) \int \frac{dy_3 dy dx d\phi}{(2\pi)^3} e^{-(x^2+y^2)/b} \sum_{N=0}^{\infty} (-1)^N xyy_3 \\
&\times \left\{ - \left[L_n(2x^2/b) + L_{n-1}(2x^2/b) \right] \left[L_N(2y^2/b) - L_{N-1}(2y^2/b) \right] \right. \\
&\times \left(\frac{(x_3 - y_3)^2 \bar{\mathcal{F}}^{(mag)}}{(x_3 - y_3)^2 + x^2 + y^2 - 2xy \cos \phi} + \frac{1}{2} \bar{\mathcal{F}}^{(el)} \right) \\
&+ \left[L_n(2x^2/b) - L_{n-1}(2x^2/b) \right] \left[L_N(2y^2/b) + L_{N-1}(2y^2/b) \right] \\
&\times \left. \left(\bar{\mathcal{F}}^{(mag)} - \frac{1}{2} \bar{\mathcal{F}}^{(el)} \right) \right\}, \tag{2.26}
\end{aligned}$$

where

$$\begin{aligned}
\bar{\mathcal{D}}^{(mag)} = & \frac{|\bar{q}|^4 \text{sign}(\mu) \text{sign}(\bar{\mathcal{M}}_N^2 - 1)}{2 \left[|\bar{q}|^6 + \left(\frac{\pi d^2}{4}\right)^2 (\bar{\mathcal{M}}_N - \text{sign}(\mu))^2 \right]} \\
& - |\bar{q}| \text{sign}(\mu) \frac{\frac{d^2}{4} (\bar{\mathcal{M}}_N - \text{sign}(\mu)) \ln \frac{|\bar{q}|^3}{\frac{\pi d^2}{4} |\bar{\mathcal{M}}_N - \text{sign}(\mu)|}}{|\bar{q}|^6 + \left(\frac{\pi d^2}{4}\right)^2 (\bar{\mathcal{M}}_N - \text{sign}(\mu))^2} \\
& - \frac{|\bar{q}|^4 \text{sign}(\mu)}{2 \left[|\bar{q}|^6 + \left(\frac{\pi d^2}{4}\right)^2 (\bar{\mathcal{M}}_N + \text{sign}(\mu))^2 \right]} \\
& + |\bar{q}| \text{sign}(\mu) \frac{\frac{d^2}{4} (\bar{\mathcal{M}}_N + \text{sign}(\mu)) \ln \frac{|\bar{q}|^3}{\frac{\pi d^2}{4} (\bar{\mathcal{M}}_N + \text{sign}(\mu))}}{|\bar{q}|^6 + \left(\frac{\pi d^2}{4}\right)^2 (\bar{\mathcal{M}}_N + \text{sign}(\mu))^2}, \tag{2.27}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{D}}^{(el)} = & \frac{\Theta[\bar{\mathcal{M}}_N^2 - 1]}{\sqrt{|\bar{q}|^2 + d^2} \left[\left(\sqrt{|\bar{q}|^2 + d^2} + \bar{\mathcal{M}}_N \right)^2 - 1 \right]} \\
& - \frac{\text{sign}(\mu) \Theta \left[1 - \bar{\mathcal{M}}_N^2 \right] \left(\sqrt{|\bar{q}|^2 + d^2} + \text{sign}(\mu) \right)}{\sqrt{|\bar{q}|^2 + d^2} \left[\left(\sqrt{|\bar{q}|^2 + d^2} + \text{sign}(\mu) \right)^2 - \bar{\mathcal{M}}_N^2 \right]}, \tag{2.28}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{F}}^{(mag)} = & \frac{1}{\bar{\mathcal{M}}} \left\{ \frac{|\bar{q}|^4 \text{sign}(\bar{\mathcal{M}}_N^2 - 1)}{2 \left[|\bar{q}|^6 + \left(\frac{\pi d^2}{4}\right)^2 (\bar{\mathcal{M}}_N - \text{sign}(\mu))^2 \right]} \right. \\
& - |\bar{q}| \frac{\frac{d^2}{4} (\bar{\mathcal{M}}_N - \text{sign}(\mu)) \ln \frac{|\bar{q}|^3}{\frac{\pi d^2}{4} |\bar{\mathcal{M}}_N - \text{sign}(\mu)|}}{|\bar{q}|^6 + \left(\frac{\pi d^2}{4}\right)^2 (\bar{\mathcal{M}}_N - \text{sign}(\mu))^2} \\
& + \frac{|\bar{q}|^4}{2 \left[|\bar{q}|^6 + \left(\frac{\pi d^2}{4}\right)^2 (\bar{\mathcal{M}}_N + \text{sign}(\mu))^2 \right]} \\
& \left. - |\bar{q}| \frac{\frac{d^2}{4} (\bar{\mathcal{M}}_N + \text{sign}(\mu)) \ln \frac{|\bar{q}|^3}{\frac{\pi d^2}{4} (\bar{\mathcal{M}}_N + \text{sign}(\mu))}}{|\bar{q}|^6 + \left(\frac{\pi d^2}{4}\right)^2 (\bar{\mathcal{M}}_N + \text{sign}(\mu))^2} \right\}, \tag{2.29}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}^{(el)} = & - \frac{\Theta[1 - \mathcal{M}_N^2]}{\sqrt{|\bar{\mathbf{q}}|^2 + d^2} \left[\left(\sqrt{|\bar{\mathbf{q}}|^2 + d^2} + 1 \right)^2 - \mathcal{M}_N^2 \right]} \\
& + \frac{\Theta \left[\mathcal{M}_N^2 - 1 \right] \left(\sqrt{|\bar{\mathbf{q}}|^2 + d^2} + \mathcal{M}_N \right)}{\mathcal{M}_N \sqrt{|\bar{\mathbf{q}}|^2 + d^2} \left[\left(\sqrt{|\bar{\mathbf{q}}|^2 + d^2} + \mathcal{M}_N \right)^2 - 1 \right]}, \tag{2.30}
\end{aligned}$$

with $\mathcal{M}_N^2 = y_3^2 + 2Nb + a^2$ and $|\bar{\mathbf{q}}|^2 = (y_3 - x_3)^2 + y^2 + x^2 - 2xy \cos \phi$.

It is instructive to note that the function under the integral in the expression for $\bar{\mu}_{5,n}$ contains an overall factor of y_3 in the numerator. Clearly, such a dependence on y_3 is not very helpful for the numerical convergence of the integral. By taking into account, however, that the rest of the integrand depends on y_3 only via $(y_3 - x_3)^2$ and y_3^2 combinations, the convergence can be substantially improved by using the following identity:

$$\begin{aligned}
& \int_{-\infty}^{\infty} dy_3 y_3 F \left((y_3 - x_3)^2, y_3^2 \right) \\
= & \int_{-\infty}^{\infty} dy_3 \frac{y_3}{2} \left[F \left((y_3 - x_3)^2, y_3^2 \right) - F \left((y_3 + x_3)^2, y_3^2 \right) \right]. \tag{2.31}
\end{aligned}$$