U.S. and Chinese Middle School Mathematics Teachers' Pedagogical Content Knowledge: The Case of Functions

by

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ABSTRACT

This study investigated the current state of the U.S. and Chinese urban middle school math teachers' pedagogical content knowledge (PCK) for the topic of functions. A comparative, descriptive case study was employed to capture the PCK of 23 teachers in Arizona and of 28 teachers in Beijing, regarding their instructional knowledge, understanding of student thinking and curricular knowledge—three key components based on Shulman's conceptualization of PCK—related to functions. Cross-case comparisons were used to analyze the PCK of teacher groups across countries and socio-economic statuses (SES), based on the questionnaire, lesson plan, and interview data.

This study finds that despite cultural differences, teachers are likely to share some commonalities with respect to their instructional decisions, understanding of student thinking and curricular knowledge. These similarities may reflect the convergence in teaching practice in the U.S. and China and the dedication the two countries make in improving math education. This study also finds the cross-country differences and cross-SES differences regarding teachers' PCK. On the one hand, the U.S. and Chinese math teachers of this study tend to diverge in valuing different forms of representations, explaining student misconceptions, and relating functions to other math topics. Teachers' own understanding of functions (and mathematics), standards, and high-stakes testing in each country significantly influence their PCK. On the other hand, teachers from the higher SES schools are more likely to show higher expectations for and stronger confidence in their students' mathematical skills compared to their counterparts from the lower SES schools. Teachers' differential beliefs in students' ability levels significantly contribute to their differences between socio-economic statuses.

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CHAPTER 1

INTRODUCTION

In Chapter 1, I introduce the present study under the "research problem" heading beginning with background information regarding my research problem, providing a brief summary of literature on the problem and leading up to my statement of the problem. The remaining sections focus on the significance of this study, the purpose of this study, the conceptual framework and the research questions.

The Research Problem

Early in the 1980s, *A Nation at Risk* (National Commission on Excellence in Education, 1983) pointed out the lack of talents with advanced mathematics and science skills in the workforce in the U.S. This report raised concerns about students' mathematics achievement and mathematics education within the U.S. Almost at the same time, China's "open-door" policy pushed educational reform, especially the reform of mathematics and science education, onto the frontlines in order to meet the needs of "Four Modernizations" (Xu & Huang, 1988). Both countries were dedicated to the improvement of mathematics education. More recently, students from East Asian countries (i.e., China, Japan, and Korea) have continuously outperformed U.S. students in mathematics (and science) on a variety of international benchmark tests, such as Programme for International Student Assessment (PISA) and Trends in International Mathematics and Science Study (TIMSS). This trend highlighted the gaps in students' mathematics achievement between the U.S. and East Asian countries, such as China

[1](National Center for Educational Statistics [NCES], 2009, 2013; The Organisation for Economic Co-operation and Development [OECD], 2009, 2013). What leads to the achievement gap in mathematics?

In response, researchers and educators in both countries have investigated this issue. Researchers pointed to a variety of attributes, such as curriculum (Cai & Watanabe, 2002; Gao & Bao, 2009; Schmidt, Cogan, Houang, & McKnight, 2011), cultural beliefs (Peng, 2007; Qiao & Tang, 2002), and teachers' impact (Cogan, Schmidt, & Wiley, 2001; Stronge, Ward, & Grant, 2011; Stronge, Ward, Tucker, & Hindman, 2007; Vandevoort, Amrein-Beardsley, & Berliner, 2004), to explain the achievement disparities evidenced in a variety of international benchmark tests. Among all these attributes, teachers' impact is regarded as one of the most important factors influencing student achievement (Cogman et al., 2001; Hill, Rowan, & Ball, 2005; Stronge et al., 2011; Tchoshanov, 2011; Vandevoort et al., 2004). It is recognized, both in China and in the U.S., that students' understanding of mathematics, their ability to solve problems of mathematics, as well as their connections between the mathematics world and real world, are all shaped by the teaching they encountered in school (An, Kulm, & Wu, 2004; Cogman et al., 2001; Gu, 2014; Moy & Peverly, 2005; Tchoshanov, 2011).

Researchers have advanced educational arguments supporting the impact of teachers' characteristics on student learning. Which teachers' characteristics lead to student learning? Studies that measure teacher characteristics tend to approach teacher and teaching primarily in three ways (Hill et al., 2005). The first large set of studies

[1] Two big cities of China -Shanghai and Hongkong -participated in the 2009 and 2012 PISA. In both 2009 and 2012, Shanghai ranked top one on the mathematics scale. Hongkong ranked fourth in 2009 and third in 2012 on the mathematics scale. In 2009, the U.S. ranked 24th on mathematics literacy in the 65 OECD countries and education systems. In 2012, the U.S. was significantly outperformed by 29 OECD nations and jurisdictions in mathematics.

investigates the relationship between teacher behaviors and student achievement, arguing what a teacher does in a classroom affects student achievement (Brophy & Good, 1986; Hiebert et al. 2003). For example, the amount of time actually spent teaching mathematics rather than engaging in class management or chatting is found to be closely related to student learning (Brophy & Good, 1986; Hiebert et al., 2003). This line of research is sometimes also called educational process-product research (Hill et al., 2005). Process-product research, however, does not pay much attention to specific content. Subject matter and how the subject matter is taught in classrooms are missing in this literature (Shulman, 1986). The second approach, also called educational production function research (Hanushek, 1996; Monk, 1989), tend to examine the relationship between educational resources and student achievement. Teachers are regarded as one of the educational inputs. Teachers' intellectual resources, which are usually measured by their own education attainment, certification status, teacher knowledge, years of teaching, and so forth, are believed to significantly influence student achievement (Hanushek, 1996, 2003; Ingersoll, 2005). Educational production function research, however, usually use teachers' performance on mathematics ability tests as well as method courses teacher take in college as indicators of teacher knowledge (Hanushek, 2003). This might be problematic because "measuring quality *teachers* through performance on tests of basic verbal or mathematics ability may overlook key elements in what produces quality *teaching* (Hill et al., 2005, p. 375, italics original). The critiques of the deficiencies of these two lines of research have led to the emergence of a third line of research with a focus on teachers' pedagogical content knowledge (PCK), which not only brings content

back to the center of teacher knowledge, but also bridges content knowledge and the practice of teaching.

Teachers' pedagogical content knowledge, as termed by Lee Shulman (1986), is a kind of knowledge that allows teachers to transform their own mathematics knowledge so that they are teaching in a way that easily comprehensible to their students in class. Or in Shulman's (1986) words, PCK is "beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching (p. 9)", such as by using different representations of specific ideas and concepts and understandings of what makes the learning of a specific topic difficult or easy for students. PCK intertwines content and pedagogy (Ball, 2000), making knowing content and instruction, and knowing content and student, the core of teacher knowledge. Studies, both in China and the U.S., consistently show the significant impact of teachers' pedagogical content knowledge on student learning and student achievement (Ball & Bass, 2003; Cankoy, 2010; Zhou, Peverly, & Xin, 2006). These researchers argue that the deficiencies in mathematics teachers' pedagogical content knowledge led to these learning gaps in mathematics (Zhou et al., 2006). Based on these findings, we have an arguably solid approach to comparing teachers in China and the U.S. – measuring their pedagogical content knowledge. My question then, is what does Chinese and the U.S. mathematics teachers' pedagogical content knowledge look like? What is the current state of Chinese and U.S. math teachers' PCK? What similarities and differences can be found?

Only a relative handful of studies (e.g., An et al., 2004; Cai, 2005; Huang & Cai, 2011; Ma, 1999) have specifically compared Chinese and U.S. mathematics teachers' pedagogical content knowledge. However, most of them have focused on merely one or

two aspects of pedagogical content knowledge, such as on the knowledge of student thinking (An et al., 2004) or on the knowledge of representations (Huang & Cai, 2011). Comprehensive comparisons of Chinese and U.S. mathematics teachers' PCK are rarely found in the literature. The most comprehensive China-U.S. comparison is Ma's (1999) study on Chinese and U.S. elementary mathematics teachers' PCK. Her research revealed sharp variances regarding Chinese and U.S. teachers' knowledge of representing mathematical ideas to students, of student misunderstandings, and of curriculum. Ma's focus, however, was on elementary mathematics teachers. Secondary mathematics teachers' PCK, especially middle school mathematics teachers' PCK, has been rarely examined through a comparative perspective.

The scarcity of information on middle school teachers' pedagogical content knowledge through a comparative perspective is regrettable since middle school mathematics is "a critical gateway to high school course taking and college enrollment" (Hill, 2007, p. 96) and "the middle school years can be a predictor of success" (Riley, 1997, p. 3). Also, comparisons on teaching middle school mathematics inform how middle school teacher preparation should be structured within a country (Schmidt et al., 2008). In China, about one-third of middle school mathematics teachers only hold an associate bachelor degree – a three-year degree (Chinese Department of Education, 2011). As Ma (1999) pointed out, a large number of Chinese teachers complete ninth grade, attend normal school for about three years, and then go to teach in elementary or middle school. Chinese math teachers usually have shorter formal education compared to their U.S. counterparts. However, in the U.S., only 41% of eighth-grade math teachers majored

in math in college, which is 30% lower than the international average (U.S. Department of Education, 2006).

This study investigates the current state of Chinese and U.S. middle school mathematics teachers' pedagogical content knowledge in the area of mathematical functions. This study is focused on middle school mathematics teachers, who are usually overlooked as a potential source of data in the China-U.S. comparative education literature. This study selects mathematical functions as the mathematical topic through which Chinese and U.S. teachers' pedagogical content knowledge is examined because function is an important concept and topic in secondary mathematics, and plays a vital role in students' entire mathematical education (Selden & Selden, 1992).

Significance of the Study

The present study is important for several reasons. First, this study contributes to the research literature, especially to the international and comparative research literature, on teachers' pedagogical content knowledge of mathematics and on mathematics education. Second, this study provides helpful information for middle school teachers in China and the U.S. because it presents comparative information on mathematics teachers' pedagogical content knowledge between China and the U.S. Third, this study informs policies on middle school mathematics teacher recruitment and training because it will provide insights into the current nature of teachers' pedagogical content knowledge for teaching mathematics in both countries. Finally, as Kaiser (1999) puts it, a comparative study can help us to better understand how to improve educational effectiveness and to enhance the understanding of our own education and society. Through comparison, this

study provides an opportunity for educators in both countries to be aware of "alternatives" used in the other country and to learn from each other.

Purpose of the Study

The purpose of this study was to investigate Chinese and U.S. middle school mathematics teachers' pedagogical content knowledge of mathematical functions. The study examined and compared teachers' pedagogical content knowledge of middle-school mathematical functions between China and the U.S. The purpose of the study was not to determine whether teachers in one country are better than in another. Instead, the purpose was descriptive – to define the current state of middle school mathematics teachers' PCK in both countries. The present study did not search for explanations as to why and how Chinese and U.S. middle school mathematics teachers acquired different (or similar) pedagogical content knowledge. And although some research (e.g., Cai, 2005) showed that teachers' knowledge of presenting and formulating mathematical topics reflects their beliefs toward mathematics and teaching, sources of teachers' pedagogical content knowledge were not the focus of the study.

Conceptual Framework

As Joseph A. Maxwell articulated in his *Qualitative Research Design*, the function of a conceptual framework is to "inform the rest of your design- to help you to assess and refine your goals, develop realistic and relevant research questions, select appropriate methods and identify potential validity threats to your conclusions" (Maxwell, 2005, pp. 33-34). The present study draws primarily upon Lee S. Shulman's theoretical perspective on teachers' pedagogical content knowledge. His categorization of teacher knowledge and conceptualization of pedagogical content knowledge inform this study,

especially in determining what research questions to be asked. Below, I describe my conceptual framework, followed by my research questions.

Shulman's perspectives on pedagogical content knowledge.

Shulman (1986, 1987) categorized teacher knowledge into seven major categories: content knowledge; general pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; and knowledge of educational ends, purposes, and values, and their philosophical and historical grounds. Among these seven categories, subject matter knowledge - content knowledge, curriculum knowledge, and pedagogical content knowledge- together comprise what Shulman (1986) called the "missing paradigm" in research and in policy. The refocus on subject matter reframes the role of content and pedagogy in teaching – questioning the generic relationships between teacher behaviors and student academic gains- and bringing content back to the central role in teaching practice. The most influential construct created by Shulman (1986, 1987), pedagogical content knowledge, is defined as:

> That special amalgam of content and pedagogy that is uniquely the providence of teachers, their own special form of professional understanding… Pedagogical content knowledge… identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, issues are organized, represented, and adapted to diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the

understanding of the content specialist from that of the pedagogue. (Shulman, 1987, p. 8)

Shulmans' definition of PCK helps to bridge content knowledge and the practice of teaching. To provide an analytical frame for distinguishing what counts as PCK and what cannot, Shulman (1986) defined teachers' pedagogical content knowledge as including:

> The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations- in a word, the ways of representing and formulating the subject that makes it comprehensible to others.
>
> An understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 9)

The first component of pedagogical content knowledge focuses on the representations of the subject to make it comprehensible to others. Teachers are actually transforming what they already understand of the subject to make it ready for effective instruction (Shulman, 1987). This "make it ready" process involves the construction of appropriate representations, of mathematical tasks, of examples and explanations, and more, all based on their understanding of a particular content. Underlying teachers' instructional decisions is their pedagogical content knowledge. From this perspective, the first component of Shulman's pedagogical content knowledge is a domain combining teachers' knowledge of content and of instruction strategies.

The second component of pedagogical content knowledge focuses on student thinking. Understanding students' conceptions and misconceptions in a specific content area is the core of teachers' knowledge of student thinking. Knowing student conceptions and misconceptions indicates that teachers need to know students' prior knowledge and incorporate students' previous learning experiences; and to be able to address students' misconceptions or misunderstandings and use various strategies to make corrections. Thus, teachers' knowledge of content and students can be viewed as the second component of Shulman's concept of pedagogical content knowledge.

Teachers' knowledge of content and instruction, and teachers' knowledge of content and students comprise the two essential components of Shulman's construct of pedagogical content knowledge, providing the lens for this study to investigate teachers' pedagogical content knowledge. However, these two components alone do not frame my conceptual framework. A third element to frame this study is curricular knowledge. Although Shulman did not initially place curricular knowledge within pedagogical content knowledge, he did point out the importance of knowing how subject matter topics were related over the span of the curriculum especially for secondary school teachers (Shulman, 1986). He referred to curricular knowledge as knowing the curriculum and its associated materials, the lateral curriculum knowledge, and the vertical curriculum knowledge. To point out the importance of knowing the curriculum and its associated materials, he articulated:

> The curriculum is represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of

characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances. (Shulman, 1986, p. 10)

This articulation indicates that teachers need to know the organization of the topics for the grades they are teaching, to understand the goals of a particular topic in the standards, and to know the materials including textbooks they can use. By lateral curriculum knowledge, Shulman (1986) referred to teachers' ability to "relate the content of a given course or lesson to topics or issues being discussed simultaneously in other classes" (p. 10). By vertical curriculum knowledge, Shulman (1986) referred to teachers' "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (p. 10). These three aspects together comprise teachers' subject-matter-specific curricular knowledge which provides an additional lens for this study to investigate teachers' pedagogical content knowledge.

The three components (or domains) of teachers' pedagogical content knowledge - teachers' knowledge of content and instruction, teachers' knowledge of content and students, and teachers' knowledge of curriculum- help me to develop three relevant research questions, respectively, to explore my research problem. Without setting essential components, it will be difficult to conduct a valid comparative study between different cultures. It is useful to take Shulman's conceptualization of pedagogical content knowledge as a frame to explore teachers' pedagogical content knowledge between the two countries. There is, however, one concern regarding the conceptual framework that I would like to address before I proceed to the specification of my research questions.

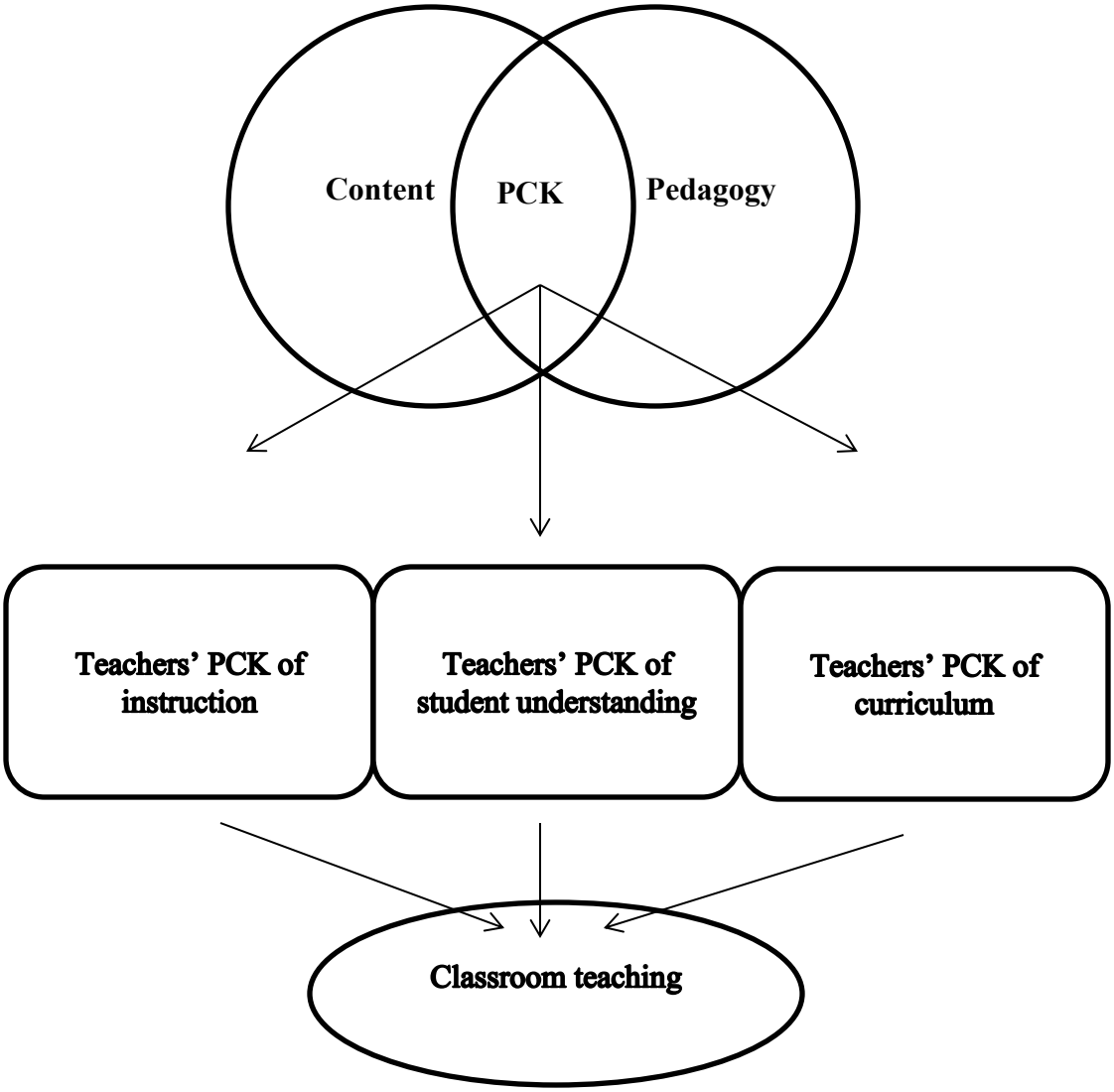


Figure 1 Shulman's (1986, 1987) perspectives on pedagogical content knowledge

As Calderhead (1987, cited in Kagan, 1990, p. 456) argued, "a theory can be used heuristically, so that a model provides an initial framework and data are allowed to interact with and modify the model". Shulman's conceptualization of teachers' pedagogical content knowledge is "not fixed and final" (Shulman, 1987, p. 12). It is developing. Although a wide variety of empirical studies have been devoted to PCK, the consensus regarding the theories and measures still have not been achieved yet and I

elaborate on this in the next chapter. The study of PCK needs more "theoretical development, analytic clarification, and empirical testing" (Ball, Thames, & Phelps, 2008). My work here provided details on Shulman's domains of pedagogical content knowledge and further validated knowledge of those domains in mathematics through cross-national comparison. This is, to a certain degree, my work is similar to what Calderhead referred to as "modifying the model" (1987, cited in Kagan, 1990, p. 456).

Research Questions

The three research questions that were explored in this study are:

1. What instructional decisions do Chinese and U.S. middle school mathematics teachers make when planning a lesson to introduce the concept of function?

2. How do Chinese and U.S. middle school mathematics teachers understand student conceptions and misconceptions of functions?

3. What curricular knowledge of the topic of functions do Chinese and U.S. middle school mathematics teachers demonstrate?

These research questions are developed within the frame of Shulman's (1986, 1987) conceptualization of pedagogical content knowledge. I explored within each of these components of teachers' pedagogical content knowledge and presented a more complete comparative picture of Chinese and U.S. middle school mathematics teachers' pedagogical content knowledge.

Key Terms

Conception: The organization of knowledge in the human mental system or in the long-term memory, which is developed at the pre-concept stage, that is, prior to learning the formalized concept. For the purpose of this study, this term refers to students'

conceptions about mathematical concepts.

Curriculum: A set of courses and content offered at schools.

Function: In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

Mathematical representation: The act of capturing a mathematical concept or relationship in some form and to the form itself. In this study, representations mainly refer to the external representation which can be understood as any visible sign, character, and object used to stand for something other than itself.

Mathematical task: A certain type of activity that teachers construct or develop for students to implement in order to learn a particular mathematical idea.

Mathematics teachers: Teachers teaching mathematics in school.

Middle school: A level of schooling between elementary school and high school. For the purpose of this study, middle school in Arizona refers to grade 7 and grade 8 (and sometimes including grade 9), while middle school in China refers to grade 7 through grade 9.

Misconception: Those mistaken ideas or views students may have as a result of previous inadequate teaching, informal learning or poor memory, leading to learning difficulties in mathematics. For the purpose of this study, this term refers to students' misconceptions about mathematical concepts.

Pedagogical Content Knowledge: A term coined by Lee Shulman (1986) that describes a kind of knowledge that teachers transform to become comprehensible for their students.

Standards: Sometimes called content standards, usually defines what students at a particular grade need to know or learn as well as what students need to be capable of doing.

CHAPTER 2

REVIEW OF RELATED LITERATURE

Since the term "pedagogical content knowledge (PCK)" was coined in the mid-1980s, different perspectives have emerged from this construct, despite being relatively new. The existing literature views teachers' pedagogical content knowledge from the perspectives of a variety of subject areas where some of the resultant themes from the varying subject areas overlap. This review is not seeking to encompass all aspects of teachers' pedagogical content knowledge. The focus of this review is mathematics teachers' pedagogical content knowledge. My purpose is to create a review summarizing current findings on mathematics teachers' pedagogical content knowledge, setting up a research background, and showing implications for the present study. In the following sub-sections, I outline the methodology for selecting which research to include, provide a synthesis of mathematics teachers' pedagogical content knowledge, summarize key findings from this research, and discuss the implications for further research.

I began my review by searching in electronic databases both in English and Chinese. I searched in educational databases such as ERIC, PsycInfo, JSTOR, and China Academic Journals Database as well as Google Scholar, using search terms including "teacher knowledge", "pedagogical content knowledge", and "mathematical knowledge for teaching". I also searched in hard-copy documents at the ASU libraries. These searches included online mathematics teachers' organizations such as the National Council of Teacher of Mathematics (NCTM), as well as government websites such as National Center for Educational Statistics (NCES) and Chinese Department of Education (CDE). In my selection of library and research publications, most works from my

selection were empirical, meta-analyses, and literature reviews in peer-reviewed journals. These resources were selected based on the methodological rigor of the studies and the frequencies with which studies were cited. Most works included in this review were published within the period of 2000 to the present providing the latest perspectives on mathematics teachers' pedagogical content knowledge. These works also pointed me to some key studies I ought to include from the 1980s and 1990s. The following themes emerged in this body of existing literature:

Theoretical development of pedagogical content knowledge.

Empirical studies on pedagogical content knowledge.

Developing measures for teachers' pedagogical content knowledge.

After reviewing studies of mathematics teachers' pedagogical content knowledge, I provide an additional review of studies of teachers' PCK on teaching mathematical functions.

What do mathematics teachers need to know about teaching functions?

Scattered Theoretical Perspectives on PCK

The concept of pedagogical content knowledge can be traced to the beginning of the last century when Dewey (1902/1983) pointed out that teachers needed to make topics and concepts in specific subject matter accessible to their students. However, the term "pedagogical content knowledge" was not coined until the mid-1980s when Lee Shulman (1986, 1987) proposed this special domain of teacher knowledge. Although the term pedagogical content knowledge has been widely used in research in the past two decades, the consensus on what is meant by pedagogical content knowledge has not been

achieved. The potential of pedagogical content knowledge has been underspecified, or thinly developed.

Looking back on Shulman and his colleagues' conceptualization of PCK.

Shulman and his colleagues based their theory on their empirical research in the *Knowledge Growth in Teaching* project in which new and experienced teachers across mathematics, science, English literature, and history subject areas were recruited to understand how new teachers learn to teach and to identify the differences of content pedagogy between novice and expert teachers (Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). Shulman and colleagues (Shulman, 1986, 1987; Wilson et al., 1987) worked deliberately across subject areas in order to provide a comparative presentation of general characteristics of teacher knowledge. Based on data from this project, they made major contributions to the theoretical development of pedagogical content knowledge. First, they brought content back to the forefront of teaching, thus highlighting the importance of subject matter knowledge and its transformation for efficient teaching. They argued that the subtleties of content pedagogy rather than general teacher behaviors such as the management of classrooms were more closely connected with teacher effectiveness and student achievement. Second, their work created a new era of studies on teacher effectiveness and teacher professional development as they provided a conceptual orientation and analytic framework. They categorized teacher knowledge and identified distinctions between categories of teacher knowledge that could matter for effective teaching. They also identified subdomains and nature of each subdomain within the pedagogical content knowledge frame. These are valuable tools to use to conduct related research.

Shulman's conceptualization of pedagogical content knowledge, however, still needs more development. First of all, Shulman's (1986, 1987) categorization of PCK and characteristics of each category as well as sources of PCK are still underspecified. Shulman (1987) actually saw the understanding of teacher knowledge as incomplete and called for more studies devoted to developing the conceptualization of pedagogical content knowledge. Secondly, they emphasized the role of content in teaching practice, arguing that content-specific categories of teacher knowledge were most closely connected to teaching effectiveness. In the meantime, they tried to find some general characteristics of the nature of pedagogical content knowledge across subjects. The dilemma brought about by their emphasis of content-specific nature and their intention of finding common characteristics across subjects led to a growing research body on teacher knowledge in different subject areas. Thirdly, Shulman (1987) viewed teachers' knowledge both as represented in their thinking, as an internal construct, and observed in their teaching practice, as an external construct. This duality thus generated a wide variety of measures to study pedagogical content knowledge.

After the introduction of the notion of pedagogical content knowledge, numerous attempts have been made to study teachers' PCK in a wide variety of subject areas. The field has experienced significant growth regarding its theoretical development, empirical studies, and measurement development in the past two decades. In the next section and following two sections, I will present how my study is situated in terms of what is currently known about PCK, conceptually, empirically and methodologically.

Theoretical development of PCK.

Despite Shulman's (1986) call for a coherent theoretical framework, much of pedagogical content knowledge has remained underdeveloped since its first conception in the mid-1980s. Here, I will review several researchers' works which have added theoretical contributions in order to show the current range of theoretical perspectives and their development within the construct of PCK. Specifically, I will primarily review Grossman's (1989, 1990) work in English literature, Cochran, DeRuiter, and King's (1993) constructivist view of PCK, Ness-Newsome's (1999) work in science, Borko's (1999) work in science, Carlsen's post-structural view of PCK (1999), and Ball, Hill and their colleagues' (1990, 2004, 2005, 2008) work in Mathematics.

Based on research of six contrasting cases of beginning English teachers, Grossman (1989, 1990) expanded on Shulman's definition, characterizing four essential components of teachers' pedagogical content knowledge: conceptions of purpose for teaching a subject at different grade levels; knowledge of student understandings, conceptions, and misconceptions of particular topics in a subject matter; curricular knowledge including knowledge of curriculum materials and knowledge about "both the horizontal and vertical curricula for a subject" (Grossman, 1990, p. 8); and knowledge of instructional strategies and representations for teaching particular topics. Grossman (1990) also identified four key sources contributing to the development of teachers' pedagogical content knowledge. The first key source was teachers' memories of how their own teachers taught in elementary and secondary school, sometimes called the "apprenticeship of observation" (Lortie, 1975). The second and third sources were teachers' subject matter knowledge and teacher education, respectively. The last source

was teachers' classroom teaching experience. These can also be regarded as an expansion of Shulman's identification of PCK sources. Grossman (1990) provided a comprehensive picture of the delineation and sources of PCK. Her work is a theoretical development as well as an empirical test of Shulman's (1986, 1987) theoretical hypotheses on the subject of English. In addition, Grossman (1989, 1990) emphasized the importance of participation and experience in teacher preparation programs in order for beginning teachers to acquire PCK, and idea which informs the design of teacher preparation programs.

Cochran, DeRuiter, and King (1993), from a constructivist perspective, proposed a modification of Shulman's concept of PCK – pedagogical content knowing (PCKg) – to emphasize knowing and understanding as active processes. Knowledge is actively created and constructed by knowers rather than passively imparted or transferred. Pedagogical content knowing is defined as "a teacher's integrated understanding of four components of pedagogy, subject matter content, student characteristics, and the environmental context of learning" (Cochran et al., 1993, p. 266). Different from Shulman's central idea of "transformation", Cochran and colleagues focused on "integration". They placed more emphasis on students' characteristics and the environmental context of learning, arguing that teachers develop their pedagogical knowledge and subject matter knowledge situated in their understanding of students and of the context of learning. The integration of the four PCKg components formed pedagogical content knowing, which is continuously developing as teachers simultaneously experience the PCKg components.

Gess-Newsome (1999) reviewed conceptual orientations of studies on PCK and summarized models of teacher cognitions into two extremes: (1) integrative models in

which PCK does not exist as a domain of knowledge, and where knowledge needed for classroom teaching derive from the integration of three independent knowledge bases: subject matter knowledge, pedagogical knowledge, and contextual knowledge; and (2) transformative models in which PCK, the only knowledge used in classroom teaching, is transformed from subject matter knowledge, pedagogical knowledge, and contextual knowledge. As she claimed, most researchers positioned themselves between the integrative and transformative models of teacher knowledge. This left them at a sort of a mid-point point, which inevitably led to a position of less theoretical power because of their weak distinctions between what can be recognized as PCK and what cannot.

Magnusson, Krajcik and Borko (1999) defined pedagogical content knowledge as "a teacher' understanding of how to help students understand specific subject matter…[it] includes knowledge of how particular subject matter topics, problems, and issues can be organized, represented, and adapted to the diverse interests and abilities of learners, and then presented for instruction" (p. 96). They viewed the defining feature of pedagogical content knowledge as lying in its conceptualization as "the result of a transformation of knowledge from other domains" (Magnuson et al., 1999, p. 96). Similar definitions can be found in Niess's (2005) research that studied the integration of technology into the science and mathematics teacher preparation program. Niess's work defined pedagogical content knowledge as "an integrated knowledge structure of teaching their specific subject matter – the intersection of knowledge of the subject matter with knowledge of teaching and learning" (p. 510). Defined in this way, however, pedagogical content knowledge seemed to include "almost everything a teacher might know in teaching a particular topic, obscuring distinctions between teacher actions, reasoning, beliefs, and

knowledge" (Ball et al., 2008, p. 394). However, Magnusson, Krajcik and Borko (1999) did bring into PCK the fifth component - knowledge of assessment in science, which added significantly to Grossman's (1990) model. This fifth component included knowledge of the dimensions of and methods to assess science learning, thus emphasizing the importance of teachers' capabilities of evaluating students' learning.

Carlsen (1999), who based his study on the science subject area, challenged the structural view prevailing in most studies on teachers' pedagogical content knowledge. He claimed that the structural view: (1) recognized and characterized knowledge domain independently from the individual; (2) failed to adequately consider the historical and cultural dimensions of knowledge; and (3) viewed knowledge as fixed and systematic. He put forward an "updated model of domains of teacher knowledge for science education" which includes science teachers' subject matter knowledge, general pedagogical knowledge, and pedagogical content knowledge with their subtleties, and he also argued that PCK is found in both a contextual and personal knowledge. Scrutiny of his model, however, reveals no evident advantage compared to Shulman' and Grossman's models. His identification of components within science teachers' pedagogical content knowledge – students' common misconceptions, specific science curricula, topic-specific instructional strategies, and purposes for teaching science, was similar to that of Grossman (1990). From this perspective, Carlsen's work can be regarded as an explication of Shulman's (1986, 1987) and Grossman's (1990) theoretical perspectives on PCK in the field of science. The most important contributions he made to the theoretical development of PCK in science were his emphases on the influence of the historical and cultural dimensions of knowledge, his advocacy of a learner-centered view

in defining PCK, as well as his rejection of dichotomous characterization of knowing and not knowing.

Ball (2000) defined mathematical pedagogical content knowledge as implementing instructional strategies based on students' academic background to present the subject matter in such a way that is comprehensible to students. Ball and her colleagues (Ball, 1990; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill, Rowan, & Ball, 2004; Hill, Shilling, & Ball, 2005) conducted a series of research inquiries with this focused question: what do teachers need to know and what are they able to do in order to teach mathematics effectively? They developed a conceptual framework under the phrase "mathematical knowledge for teaching" which was broader than the concept of mathematical pedagogical content knowledge, referring to the mathematical knowledge needed to carry out the work of teaching mathematics. Their "mathematical knowledge for teaching" structure includes six domains. The first domain is common content knowledge (CCK) - the mathematical knowledge and skill used in settings other than teaching. The second domain is specialized content knowledge (SCK), the distinct mathematical knowledge and skills unique to teaching. These two domains, together with horizon knowledge – "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2005, p. 403) – comprise their subject matter knowledge's frame. The fourth domain is knowledge of content and student (KCS) which combines knowing about students and knowing about mathematics. This domain is similar to Shulman's (1986) idea that teachers need to know students' preconceptions and misconceptions on a specific topic. The fifth domain is knowledge of content and teaching (KCT) which combines knowing about teaching and

knowing about mathematics. This domain is similar to Shulman's (1986) idea that teachers need to represent and formulate their understanding of content to instruction. These two domains, as they claimed, "coincide with the two central dimensions of pedagogical content knowledge identified by Shulman" (Ball et al., 2005, p. 402). KCS, KCT, together with knowledge of content and curriculum, comprise their frame of pedagogical content knowledge.

Ball and her colleagues' (Ball, 1990; Ball et al., 2005, 2008; Hill et al., 2004, 2005) work was important because of their contributions to the theoretical development of PCK. First of all, their conceptual framework was practice-based, using what they called a "bottom up approach". Thus, the framework they posited had a solid empirical support and bridges teachers' knowledge and practice. Second, they provided a comprehensive frame of teacher knowledge with specification of domains and characteristics. They had specific descriptions of subtleties under each domain, which they called "detailed job description". It helped to distinguish between knowledge domains. Third, the measures they created for each domain of teacher knowledge were tested on nationally representative samples, which enhanced the generalizability of their conceptual framework.

Collective and continuous efforts on the theoretical development of PCK.

The theoretical perspectives of PCK discussed above represent general development trends in the studies of PCK across different subjects. It is arguably obvious that the field has not achieved a consensus on the definition of the construct of pedagogical content knowledge and on the specifications of the components within this construct, especially when it comes to different subjects. First, researchers conceptualize

the construct of pedagogical content knowledge from a variety of paradigms such as structural, constructivist, and post-structural view, which precludes a consensus on the definition and delineation of PCK. The PCK construct has fuzzy boundaries, demanding more analytical clarity. Much still needs to be done to realize Shulman's initial charge of developing a coherent theoretical framework for content knowledge for teaching. Second, it is particularly hard for researchers to achieve a consensus on the subtleties of PCK since researchers usually work from within their own subject areas. The specification of PCK for mathematics teachers might be different from that for English teachers or for history teachers. However, it is not impossible to think that theoretical development from a particular subject can inform work in other subjects. It is also not impossible for ideas on PCK from within and across subjects to converge and form a more coherent framework. The development of the construct of PCK is a collective and continuous cultivation. More studies are needed to test Shulman's and other researchers' theoretical hypotheses and analytical tools on PCK, on the one hand, and to bring new theoretical perspectives and concrete version of knowledge into the PCK studies, on the other.

Mapping Teachers' Pedagogical Content Knowledge Empirically

Empirical studies have been devoted to studying teachers' pedagogical content knowledge from a range of subjects since the construct of PCK. My review of empirical research will now follow the three themes with a focus on the subject of mathematics. The first theme is focused on how teachers' understanding of specific content determines their PCK. The second is focused on the documentation of the state (especially inadequacies) of mathematics teachers' PCK. The third is cross-national differences between mathematics teachers' PCK. These three themes are used to help understand

what we currently know about mathematics teachers' PCK. It is noteworthy, however, that research under their respective themes is not mutually exclusive and the boundaries between them are artificial.

Content knowledge as foundation of PCK.

Mathematics teachers' PCK is believed to be significantly influenced or even determined by their understanding of specific content (Ball et al., 2008; Capraro, 2005; Hill et al., 2004; Krauss et al., 2008), and this same subject-specific content knowledge then usually plays a central role in developing mathematics teachers' PCK (Ball et al., 2008; Hill et al., 2004).

In a mixed method study where 193 student teachers from a large south-western U.S. public university were participants, Capraro and his colleagues (2008) found that mathematically competent student teachers exhibited more pedagogical content knowledge "as they were exposed to mathematics pedagogy during their mathematics methods course (Capraro et al., 2008, p. 102). This finding coincided with some of the findings of another research study conducted in Germany (Krauss et al., 2008). Krauss and colleagues (2008) conducted quasi-experimental research on 198 10^{th}-grade mathematics teachers- a nationally representative sample as claimed by the authors - in order to understand the correlation between teachers' PCK and CK and whether this interrelatedness was a function of the degree of teacher expertise. They did find a high interrelatedness between PCK and CK, but only in teachers with high expertise level, i.e. the higher a teacher's expertise, the stronger the connection between the teacher's PCK and CK in mathematics. It is noteworthy that although both of these research studies

found the correlation between PCK and CK using quantitative approaches, they did not make explicit claims that CK was the predictor of PCK.

Hill, Schilling, and Ball (2004) used factor analysis to test measures they designed for teachers' content knowledge of teaching elementary mathematics. They found a pattern that teachers' knowledge of student and content (KSC) items consistently loaded on the common knowledge of content (CKC). They suggested that mathematics content knowledge should be related to knowledge of student and mathematics. This finding was also supported by Ball and her colleagues' later research (2008). Using a qualitative approach, they argued that mathematics teachers' content knowledge played an essential role in the effectiveness of teaching (Ball et al., 2008). They contended that CK was "immensely important" to effective teaching as well as PCK. Additionally, they also argued that PCK and CK cannot be separated during the act of teaching (Ball et al., 2008).

I presented the interrelatedness of PCK and CK in order to emphasize that PCK is subject-matter specific knowledge. It is difficult for a teacher to demonstrate solid PCK without having solid CK. In the next section, I will review the current state of mathematics teachers' PCK.

Current state of mathematics teachers' PCK.

One of the most important goals of investigating the current state of mathematics teachers' PCK is to identify what inadequacies exist in order to inform what we can be done to address those inadequacies. Below, I review research on the state of mathematics teachers' PCK (inadequacies in particular) from three specific aspects of PCK. I also include research focusing on the novice-expert gaps in PCK mainly because the

inadequacies in PCK are usually found among novice math teachers in comparison to experienced math teachers.

State of mathematics teachers' PCK from three aspects.

There are three aspects that define the nature of mathematics teachers' PCK. These are (1) teachers' knowledge of content and instruction - knowing how to represent mathematics ideas for student learning, (2) teachers' understandings of student thinking, and (3) teachers' understandings of mathematics curriculum.

Representing mathematical ideas.

Representing mathematical ideas (and concepts) for students occupies the core of teachers' knowledge of content and instruction (Ball et al., 2008). In the past two decades, research has documented mathematics teachers' abilities of generating appropriate representations that make the content comprehensible for students. Constructing mathematical tasks (e.g., Arbaugh & Brown, 2005; Ball et al., 2008; Henningsen & Stein, 1997; Stein & Lane, 1996; Stein, Grover, & Henningsen, 1996; Stein, Henningsen, & Silver, 2000) and using appropriate representations (An, Kulm, & Wu, 2004; Borko et al., 1992; Confrey, Piliero, Rizzuti, & Smith, 1994; Izsak & Sherin, 2003; Huang & Cai, 2011; Ma, 1999; Ozmantar, Akkoc, Bingolbali, Demir, & Ergene, 2010) are two of the important elements on which researchers placed great emphasis on when studying teachers' knowledge of content and instruction.

Knowing how to construct mathematical tasks is important in teachers' pedagogical content knowledge. To put it briefly, a mathematical task is a certain type of activity that teachers construct or develop for students to implement in order to learn a particular mathematical idea. Mathematical tasks are central to students' learning, thus

central to teaching (Henningsen & Stein, 1997). This is not only because mathematical tasks convey messages about what doing mathematics entails (NCTM, 2000), but also because of its power "to evoke a mathematical response from the learner" (Fletcher, 1964, cited in Watson & Mason, 2007, p. 206).

Stein and her colleagues (1996, 2000) have conducted a series of research on mathematical tasks and the relationship between mathematical tasks and student learning based on their 3-year QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) project on middle school mathematics education (Henningsen & Stein, 1997; Stein & Lane, 1996; Stein, Grover, & Henningsen, 1996; Stein, Henningsen, & Silver, 2000). They found that how a teacher set up a mathematical task directly influenced the level of the mathematical task. Furthermore, the level of a mathematical task was closely related to students' implementation of the mathematical task and student learning (Henningsen & Stein, 1997; Stein & Lane, 1996; Stein, Grover, & Henningsen, 1996). Teachers integrate their goals, knowledge of subject matter, and knowledge of students to set up or construct a mathematical task (Stein & Lane, 1996). Thus, the central role of mathematical tasks makes it a good research instrument to measure teachers' pedagogical content knowledge (Arbaugh & Brown, 2005).

Mathematical tasks can be examined, as Stein and her colleagues (Henningsen & Stein, 1997; Stein & Lane, 1996; Stein, Grover, & Henningsen, 1996) illustrated, in terms of two interrelated dimensions: task features and cognitive demands. Task features include three sub-dimensions: (1) the extent to which the task is open to multiple solution strategies; (2) the extent to which the task encourages multiple representations; (3) and the extent to which the task demands explanations from the students (Stein & Lane,

1996). Cognitive demands include two levels: (1) the high level of cognitive demands which involves the use of procedures with connections to concepts, meaning, and/or understanding as well as doing mathematics; and (2) the low level which involves the use of procedures without connections to concepts, meaning and/or understanding usually based in memorization. Using this framework, they found that a mathematical task with a high level of cognitive demand produced the most student learning gains. However, it was difficult for teachers to set up and maintain a high level mathematical task – the percentage of consistency between setting up and implementing high level mathematical tasks was only 42% - in this experimental study (Stein, Grover, & Henningsen, 1996). The mathematical task framework by Stein and her colleagues (1996) was used by other researchers, such as Arbaugh and Brown (2005), to improve teachers' PCK. In their study, Arbaugh and Brown (2005) documented how the LCD (levels of cognitive demands) study helped seven high school geometry teachers, initially deemed inadequate in constructing high-level mathematical tasks, and their improvement through the course of the study.

Representation is an important construct in the research on mathematics teaching and learning because it is an important element in teachers' knowledge of content and instruction. According to the NCTM (2000), mathematical representations refer both "to process and to product- in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" (2000, p. 67). Generating appropriate representations is one of the central teaching tasks for mathematics teachers (Ball et al., 2008; Ma, 1999; Shulman, 1986, 1987; Grossman, 1990). Research consistently reveals teachers' lack of capability to produce appropriate representations for student learning

(An et al., 2004; Borko et al., 1992; Confrey, Piliero, Rizzuti, & Smith, 1994; Izsak & Sherin, 2003; Huang & Cai, 2011; Ma, 1999; Ozmantar, Akkoc, Bingolbali, Demir, & Ergene, 2010).

Ms. Daniels, a middle school math teacher in the study by Borko and her colleagues (1992), consistently showed her incompetence in constructing coherent explanations and powerful representation of the invert-and-multiply algorithm in the division of fractions for her students (Borko et al., 1992). She was "confused about" the role representations could play in "developing students' understanding of the invert-and-multiply algorithm" (Borko et al., 1992, p. 207). Similar findings were found in Ma's (1999) comparative study on Chinese and U.S. math teachers' mathematical knowledge for teaching. When asked to come up with stories or models to represent 1¼ ÷ ½, almost all the 23 U.S. teachers failed to come up with a representation of division by fractions. Only one teacher provided a "conceptually correct but pedagogically problematic representation" while all the other teachers either were unable to create a story or "made up stories with misconceptions" (Ma, 1999, p. 64).

Not being able to use multiple representations simultaneously for mathematical concepts is another inadequacy haunting many mathematics teachers. For example, in Huang and Cai's (2011) comparative study of teachers' pedagogical representations of linear functions, Chinese math teachers were found to prefer to use "a few selective representations hierarchically" (p. 160) – the use of symbolic representations to understand the concept of linear equations and graphical representations to understand the graphs of linear equations separately – compared to their U.S. counterparts. Contrasting cases were found in An and colleagues' (2004) study in which Chinese

teachers were able to use two representations – area and repeated addition – flexibly, to illustrate fraction multiplication, while their U.S. counterparts relied only on area representations to illustrate fraction multiplication. Although there are contradictory findings regarding Chinese and U.S. teachers' use of multiple representation, the core message in their studies is that teachers lose the opportunity to help their students build a complete picture of the representations of mathematical concepts when they use only one or two representations in the classroom (An et al., 2004; Huang & Cai, 2011).

In addition to the more traditional forms of representation, classrooms have introduced the use of technology as another means to represent mathematical concepts. The goal is to help students become comfortable (and flexible) with different forms of representations (Confrey et al., 1994; Ozmantar et al., 2010). Ozmantar and his colleagues' (2010) found that very few pre-service mathematics teachers in their study were able to give complete examples of "graphical", "tabular" and "algebraic" representations to explain the concept of derivative at a point, using software in a teacher preparation course - Method for Teaching Mathematics II and Technology-Aided Mathematics Teaching - in Turkey. Teachers' difficulty in using multiple representations in a rich-technology environment is troublesome for mathematics teaching nowadays.

Understanding student thinking.

Understanding students' thinking especially their conceptions and misconceptions is another central task for mathematics teachers. Research shows that how teachers construct a lesson is highly related to how teachers view their students' understanding (Thompson, 1984; Anderson & Hoffmeister, 2007). It is important for teachers to distinguish their own understanding from their students' understanding of content

(Anderson & Hoffmeister, 2007). Among researchers focusing on teachers' knowledge of student thinking, Carpenter and his colleagues (e.g., Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, Franke, Carpenter, & Carey, 1993) have done a series of remarkable experimental studies looking at teachers' knowledge of student thinking.

In their investigation of first-grade math teachers' knowledge of student thinking especially students' strategies for solving problems on addition and subtraction, Carpenter and his colleagues (1988) found that the teachers' difficulties in categorizing and rating the difficulty level of problems were mainly attributed to their failure to understand children's strategies for solving problems. For example, teachers' consistent overestimation of the difficulty of join-change-unknown problems was attributed to their misunderstanding of students' direct modeling and counting strategies for solving this kind of problem (Carpenter et al., 1988).They further pointed out the importance of understanding students' thinking and building on students' existing knowledge when teaching in the classroom (Carpenter et al., 1989). In their 1989 study, they gave the experimental group a four-week Cognitively Guided Instruction (CGI) treatment in which teachers were exposed to an extensive learning of student thinking, while the control group was given two 2-hour workshops on non-routine mathematical problems. Their findings revealed significant differences between the CGI and control group with respect to their knowledge of students' strategies for problem solving and of students' existing knowledge (Carpenter et al., 1989). Compared to their CGI treatment counterpart, teachers in the control group showed the incapability to assess the problem-solving processes of a number of students.

Carpenter and his colleagues' (Carpenter et al., 1988; Carpenter et al., 1989) experimental study not only revealed teachers' inadequacies in knowing student thinking, but also provided a solution – CGI – to improve teachers' knowledge of student thinking. Research on teachers' knowledge of students' mathematical conceptions and misconceptions has flourished in the past two decades since the presentation of CGI. Remarkable progress has been made to help integrate knowledge about children's thinking into teachers' instructional practices (e.g., Franke & Kazemi, 2001; Jansen & Spitzer, 2009; Steinberg, Empson, & Carpenter, 2004). For instance, Ms. Statz, the participant in Steinberg, Empson and Carpenter's (2004) CGI project, showed dramatic change in her knowledge of children's thinking. From Phase 1, in which Ms. Statz rarely referred to student thinking in the classroom, to Phase 4, she was able to not only integrate discussing students' thinking into her routine instruction, but also use her students' thinking to help them advance.

Understanding curriculum.

Understanding curriculum is important for teaching mathematic. A sound pedagogical content knowledge is always connected with a solid understanding of curriculum, such as the goals of curriculum, the curricular materials, the span of a particular topic in the previous and future grades, and connection to other subjects in a particular grade (Ball et al., 2008; Grossman, 1990; Shulman, 1986, 1987). However, studies consistently reveal teachers' difficulties in understanding curriculum in-depth or in connecting curriculum with their instruction, especially within the standards-based curriculum frame (Ball, 1990; Choppin, 2009; Ma, 1999; Manouchehri & Goodman, 1998; Si & Li, 2012; Stein & Kaufman, 2010).

One important aspect regarding teachers' understanding of curriculum involves recognizing the various curricular options in the mathematics content domain (Shulman, 1986). Teachers sometimes find it challenging to "flexibly select and apply the resources in those curricula to develop a comprehensive and connected set of learning experiences" (Choppin, 2009, p. 290). For example, Manouchehri and Goodman's (1998) study revealed that a sample of middle school math teachers in Missouri, especially beginning teachers, found it difficult to implement standards-based curricula. The beginning teachers relied on the standards-based materials. However, although they tended to use new textbooks, their approach of using new materials remained superficial as they used did not lead to any discussion on their mathematical significance (Manouchehri & Goodman, 1998). This is supported by Stein and Kaufma (2010) who studied mathematics teachers' use of different materials to teach scale in two school districts. They found that the level of complexity for how teachers used the curricular materials was significantly related to the lesson quality.

Teachers' curricular knowledge also involves teachers' understanding of how a topic is structured in a curriculum and how a concept is connected to other concepts. A profound understanding of curriculum requires teachers to demonstrate an understanding of longitudinal coherence in mathematics learning, which Ma (1999) called "knowledge packages" (p. 113). In her China-U.S. comparative study on elementary math teachers, Ma (1999) pointed out that compared to their U.S. counterparts, the Chinese teachers were able to tie together five important concepts- meaning of multiplication, models of division by whole numbers, concept of a fraction, concept of a whole, and the meaning of multiplication with whole numbers – to address the meaning of division by fraction. In

contrast, this knowledge package was absent from the U.S. math teachers' curricular knowledge and, in addition, this major deficiency in connecting concepts and topics was especially notable when a new idea or topic was introduced (Ma, 1999). This echoed Ball's (1990) study on prospective elementary and secondary mathematics teachers' understanding of division, in which she pointed out that these teachers only had fragmented understandings of mathematics knowledge points and they also "tended to be both rule-bound and compartmentalized" (p. 141). Later research (Si & Li, 2012) in Chinese high school math teachers' understanding of curriculum showed that teachers, especially teachers with fewer years of teaching, felt inadequate in the knowledge of how core mathematical concepts were structured within the span of high school math curriculum. It is obvious that it is challenging for teachers to connect mathematical concepts within the context of math curriculum.

In this section, I have reviewed empirical studies to reveal the current state, especially the inadequacies, of mathematics teachers' PCK from three aspects: representing mathematical ideas, understanding student thinking, and understanding of curriculum. Another way to address mathematics teachers' inadequacies in PCK is using the novice-expert comparison, which consistently reveals preservice and novice teachers' inadequacies in PCK compared to their experienced counterparts.

Novice-expert gaps in mathematics teachers' PCK.

Novice-expert differences in teacher knowledge (or expertise) are widely recognized across subject areas (Berliner, 1986; Borko et al., 1992; Cankoy, 2010; Carter, Sabers, Cushing, Pinnegar, & Berliner, 1987; Grossman, 1990; Livingston & Borko, 1990; Shanghai Qingpu Experiment Institute, 2007). In the subject of mathematics, the

novice-expert gaps in teacher knowledge, especially in pedagogical content knowledge are consistently present in empirical studies (Livingston & Borko, 1990).

Expert teachers overall are better at setting up goals, selecting appropriate instructional strategies and representing ideas, as well as giving reasoning under those strategies and representations (e.g., Cankoy, 2010; Leinhardt, 1989; Shanghai Qingpu Experiment Institute, 2007). In Leinhardt's (1989) study, the expert teachers, compared to their novice counterparts, gave better explanations, used well-known representations to explain new materials, used the same representation for multiple explanations, and incorporated prior skills of the students. Investigating mathematics teachers' topic-specific pedagogical content knowledge in the context of teaching a^0, 0! and $a \div 0$, Cankoy (2010) found that "experienced teachers suggested more conceptually and qualitative reasoning based instructional strategies for teaching of the mathematical cases than the novice teachers" (p. 761). Similar findings are evidenced in China. Case studies on two 3rd grade mathematics teachers' pedagogical content knowledge in Shanghai revealed that the expert teacher did better than the novice teacher regarding setting up goals for teaching, constructing mathematical tasks, and utilizing instructional strategies (Shanghai Qingpu Experiment Institute, 2007). For example, in setting up their goals for teaching the area of a rectangle, the novice teacher only provided a concise illustration while the expert teacher gave a detailed description of what students have learned, such as the perimeter of a rectangle, and will learn from the current lesson (Shanghai Qingpu Experiment Institute, 2007).

Novice-expert gaps were also observed regarding teachers' understandings of student thinking and responses to student questions (e.g., Li, Ni, & Xiao, 2006; Livington

& Borko, 1990). Livingston and Borko (1990) found that novice mathematics teachers experienced difficulty in responding to unexpected student questions and in providing a comprehensive picture to student questions when teaching high school review lessons. Furthermore, the expert teacher asked more explorative questions to enhance students' understanding while the novice teacher asked more memory questions (Livingston & Borko, 1990). Li, Ni and Xiao (2006) conducted a survey study on 32 Chinese elementary math teachers in China with a focus on novice-expert differences in understanding students' thinking. Their findings revealed that although novice teachers pointed out students' errors, they were not able to give the reasoning underlying these errors as well as to connect students' preconceptions of areas (Li et al., 2006).

It is obvious that most inadequacies found in mathematics teachers' PCK so far are associated with novice teachers. Much more emphasis has been placed on addressing novice teachers' inadequacies and improvement, while experienced teachers are usually absent from the picture. Or experienced teachers are contrasted as "models" when it comes to the novice-expert comparison. It is, however, not the whole story. Experienced teachers' current state of PCK needs to be addressed in a more in-depth and comprehensive way, too.

Cross-national comparisons of mathematics teachers' PCK.

In today's global society where countries are increasingly more interconnected, it is important to engage in international comparisons. It is increasingly important to see teachers' similarities and differences, and to learn from each other, through these comparisons. For example, international comparisons provide lens to see how prospective math teachers' knowledge vary across countries (Schmidt et al., 2008; Konig et al., 2011).

It then can also provide lens to investigate experienced teachers' knowledge across counties. In this section, I will first review some influential comparative studies on mathematics teachers' knowledge across countries including China. Then I will turn to some important China-U.S. comparative studies on mathematics teachers' PCK, which sheds further light on the present study.

Cross-national studies on teachers' knowledge.

Cross-national studies such as the PISA (NCES, 2011) and TIMSS (Hiebert & Stigler, 2000) paved the way for understanding country differences in student achievement and teacher knowledge in mathematics. Remarkable studies on mathematics teachers' knowledge, or "professional competence" (Schmidt et al., 2008) have been conducted by several researchers such as Hiebert and Stigler and their colleagues (Hiebert & Stigler, 2000; Hiebert et al., 2003; Kawanaka & Stigler, 1999; Stigler, Lee, & Stevenson, 1987) and Schmidt and his colleagues (e.g., Schmidt et al., 2008; Konig, Blomeke, Paine, Schmidt, & Hsieh, 2011).

Hiebert and Stigler and their colleagues (Hiebert & Stigler, 2000; Hiebert et al., 2003) based most of their comparative studies on the TIMSS video study, such as the third TIMSS Video Study of Teaching (e.g.,). An important contribution they made to the understanding of teacher knowledge across countries is how teachers from different countries structure math lessons (Hiebert et al. 2003). For example, teachers in Japan spent an average of 15 minutes on each independent mathematics problem, while other countries, such as the U.S. and Australia, only spent an average of 2 to 5 minutes on each independent mathematics problem (Hiebert et al., 2003). Furthermore, the time teachers spent on different lesson segments varied across countries. Japanese teachers spent 60

percent of a lesson on introducing the new content, much higher than did teachers in other countries. The Czech Republic and the U.S. teachers spent 58 percent and 53 percent, respectively, of a lesson reviewing. Teachers from Hong Kong spent 37 percent of the lesson time practicing new content, higher than teachers from all the other countries (Hiebert et al., 2003)

Schmidt and his colleagues' (2008) work were based on two projects: Mathematics Teaching in the 21st Century (MT21) in Bulgaria, Taiwan, Germany, Korea, Mexico, and the United States, and the Teacher Education and Development Study in Mathematics (TEDS-M) (Konig et al., 2011). Both studies focused on future math teachers in teacher preparation programs in these different countries. In their 2008 study, they designed 76 survey items to elicit information regarding future teachers' time allocation on their Teachers' Opportunities to Learn (TOL) scales and found a large amount of variation across these future teachers on most of their OTL scales (Schmidt et al., 2008). Later in their 2011 study, they found a significant difference in future middle school teachers' general pedagogical knowledge. The U.S. future middle school teachers were significantly outperformed by their counterparts in Germany and Taiwan regarding their overall general pedagogical knowledge (GPK) test scores (Konig et al., 2011).

Although these studies used different instruments (video study versus survey study) to explore teacher knowledge across countries, it is noteworthy that these cross-national comparisons share some commonalities. First of all, they all place considerable emphasis on the general characteristics of effective teaching and teachers' pedagogical knowledge (such as time allocation). Second, all the characteristics of teacher knowledge are quantified (e.g., percentages of time and scores on scales) in these large-scale studies.

Third, specific content is absent in these studies. More in-depth content-specific studies are needed to augment cross-national studies in teacher knowledge.

China-U.S. comparisons on teachers' PCK.

The most important China-U.S. comparative study of PCK is Liping Ma's in-depth investigation into Chinese and U.S. elementary teachers' knowledge of mathematics (Ma, 1999). Although she used the term "teachers' understanding of fundamental mathematics" rather than pedagogical content knowledge, her conception of mathematical understanding was "profoundly pedagogical," as it emphasized "those aspects of knowledge most likely to contribute to a teacher's ability to explain important mathematical ideas to students" (Shulman, 1999, in Ma, 1999, xi). In her study, Ma (1999) recruited 23 U.S. elementary math teachers (12 beginning and 11 experienced), who were considered "better than average" (p.xxi), and 72 Chinese elementary math teachers (40 beginning and 32 experienced teachers from a range of schools from low quality to high quality), who ranged from high to low quality. She used four topics – subtraction with regrouping, multidigit multiplication, division by fractions, perimeter, and area of a closed figure – to explore how Chinese and U.S. elementary math teachers taught similar topics differently. She examined how teachers in each country responded to a student's mistake, generated a representation of a certain topic, and responded to a novel idea raised by a student, respectively. A striking contrast in the knowledge was found between the Chinese and U.S. elementary math teachers. Almost all 72 Chinese teachers demonstrated algorithmic competence and conceptual understanding of all four topics, while the 23 U.S. teachers focused on procedures and had difficulties in division by fractions and perimeter and areas of a rectangle. Overall, the U.S. teachers' knowledge of

mathematics was fragmented compared to their Chinese counterparts, whose knowledge structure was clearly coherent. Ma's (1999) research was groundbreaking and inspiring for a number of reasons: (1) for its comparative perspective on teachers' PCK between China and the U.S., (2) for a comprehensive picture on elementary math teachers' knowledge in two countries, and (3) for her analysis of the profound distinctions between the Chinese and U.S. math teachers.

Another series of China-U.S. comparative studies can be found in Cai and his colleagues' work (Cai, 2005; Cai & Wang, 2006; Huang & Cai, 2011), focused on teachers' conceptions and constructions of mathematical representations. Based on eleven U.S. and nine Chinese elementary mathematics teachers' understanding of arithmetic average, ratio and proportion, Cai (2005) and Cai and Wang (2006) found that the U.S. and Chinese teachers had different cultural beliefs about the teaching and learning of mathematics, as well as different cultural values about representations; Chinese teachers made more detailed lesson plan than did U.S. teachers; and Chinese teachers expected students to learn more generalized strategies and representations while the U.S. teacher expected students to "solve a problem no matter what strategies or representations they use" (Cai, 2005, p. 154). In later research, Huang and Cai (2011) reported additional interesting data on Chinese and U.S. math teachers' pedagogical representations of linear relations. They found that the U.S. teachers were more likely to use multiple representations simultaneously while the Chinese teachers were more likely to use one or two representations. Numerical and tabular representations were de-emphasized by Chinese teachers compared to their U.S. counterparts.

An and her colleagues have done a series of studies on Chinese and U.S. mathematics teachers' pedagogical content knowledge with a focus on their understanding of student thinking (An et al., 2004; An, Kulm, Wu, Ma, & Wang, 2006). In their study with 28 U.S. mathematics teachers and 22 Chinese mathematics teachers at the middle school level, they found significant differences between Chinese and U.S. teachers' pedagogical content knowledge of building on students' math ideas, of addressing students' misconceptions, of engaging students in math learning, and of promoting students' thinking about mathematics in the area of fraction, ratio, and proportion. For example, when dealing with students' misconceptions, the U.S. teacher tended to use experience with a variety of models to provide concrete representations of abstract mathematical ideas while the Chinese teachers tended to use a variety of activities with a focus on "developing the explicit connection between the various models and abstract thinking" (An et al., 2004, p. 161).

Cross-national studies, especially China-U.S. comparative studies on teachers' PCK, cast light on the present study. It is, however, noted that the China-U.S. comparisons on mathematics teachers' PCK have not constituted a complete picture yet. A large amount of research (e.g., An et al., 2004; An, Kulm, Wu, Ma, & Wang, 2006; Cai, 2005; Cai & Wang, 2006; Huang & Cai, 2011; Wang & Lin, 2005) is focused on only one aspect of mathematics teachers' PCK. Ma's (1999) study provided a more complete picture of China and the U.S. mathematics teachers' PCK but her focus is on elementary teachers. Middle school teachers' PCK has not been comprehensively explored in China-U.S. comparisons. A comprehensive analysis of middle school mathematics'

pedagogical content knowledge is needed to complement the current China-U.S. comparative studies on PCK.

Looking ahead.

This review of empirical studies on mathematics teachers' PCK found a large portion focused on novice (and prospective) mathematics teachers' PCK. Experienced math teachers' PCK and their inadequacies in PCK are rarely examined and discussed. Cross-national comparisons of teachers' PCK provide a lens to investigate teachers' pedagogical content knowledge across borders. However, China-U.S. comparisons are relatively scarce in the literature and most of them focus on elementary math teachers. Consequently, more comparative studies on China and U.S. middle school mathematics teachers are needed in order to understand their knowledge and to inform middle school teacher preparation and training policies within each country.

Is There a Best Way to Measure Pedagogical Content Knowledge?

A wide variety of empirical studies have developed an array of measures to study pedagogical content knowledge since Shulman (1986, 1987) coined the term. However, no consensus has been achieved regarding the best way to measure teachers' pedagogical content knowledge. This is partly attributed to how researchers determine what counts as pedagogical content knowledge. The breadth and complexity of Shulman's (1986, 1987) original definition of pedagogical content knowledge precludes consensus among researchers regarding what exactly constitutes pedagogical content knowledge. Thus, teachers' pedagogical content knowledge has been viewed as an external construct which can be observed in teaching practice, or as an internal construct which can be elicited from teachers' thinking (Shulman, 1987).

PCK as an external construct.

Researchers usually use classroom observations as one of the most important instruments to measure teachers' pedagogical content knowledge. This practice stems from the idea that teachers' actions are more accurate representations of their knowledge (Capraro, Capraro, Parker, Kulm, & Raulerson, 2005; Huang & Cai, 2011; Livingston & Borko, 1990; Berk & Hiebert, 2009). For instance, in order to understand the novice-expert gap when constructing mathematics review lessons, Livingston and Borko (1990) observed two novice and two expert teachers teaching analytic geometry and calculus in a suburban county school system for a semester. These classroom observations allowed them to see teachers' knowledge translated into classroom practice.

Studies focused on teacher knowledge improvement often incorporated observations into their design because classroom observations provided opportunities to "see" what happened over time (Berk & Hiebert, 2009). In order to assess if and how the prospective teachers' understanding of mathematics knowledge improved in a specific teacher preparation program, Berk and Hiebert (2009) observed these prospective teachers in the classroom and collected their responses to student questions during each lesson. Using observation, Berk and Hiebert (2009) were able to both track these prospective teachers' actual responses in class and overall improvement throughout the program.

There is no definite rule declaring that classroom observation should be in line with the qualitative tradition or with the quantitative tradition. Researchers can go with either orientation. For example, Livingston and Borko's (1990) research was an in-depth qualitative piece while Berk and Hiebert's (2009) was a quantitative piece.

Critiques of observations during classroom instruction (e.g., Baxter & Lederman, 1999) pointed out that relying on teachers' classroom instruction to measure their knowledge is problematic because "myriad factors influence classroom instruction and student understanding…the level of consistency of between teachers' observed behavior and their knowledge and beliefs is highly variable (p. 158)". An over reliance on classroom observation may present a "distorted view of teachers' knowledge (Baxter & Lederman, 1999, p. 158).

PCK as an internal construct.

The limitations of observations pushed researchers to think of other measures with which to study teachers' pedagogical content knowledge. Viewing PCK as an internal construct asserts the need to access teachers' pedagogical content knowledge via their thinking. A popular approach to elicit teachers' thoughts is to have them articulate their knowledge (Kagan, 1990). Measures used to help teachers "articulate" their knowledge include surveys (and tests) (Delaney, Ball, Hill, Schilling, & Zopf, 2008; Hill et al., 2005; Hill, Schilling, & Ball, 2004; Krauss et al., 2008; Strawhecker, 2005; Zhao, Ma, Li, & Xie, 2010); questionnaires (An et al., 2004; Borko et al., 1992; Isiksal & Cakiroglu, 2010; Li et al., 2006; Turnuklu & Yesildere, 2007); interviews (Ball, 1990; Foote, 2009; Koirala, Davis, & Johnson, 2008); reflective journals (Capraro et al., 2005); and lesson plans (Cai, 2005; Cai & Wang, 2006).

Surveys and tests.

Surveys and tests have become popular in the field of PCK. This is because of their ability to study a large sample sizes which are mainly used to generalize to the larger population. Studies using survey as the main instrument usually utilize quantitative

strategies (Hill, Rowan, & Ball, 2004; Hill, Shilling, & Ball, 2005; Li et al., 2006; Zhao et al., 2010) or mixed method strategies (Capraro et al., 2005).

One study that utilized surveys to produce a groundbreaking instrument was done by Hill and her colleagues (Hill, Rowan, & Ball, 2004; Hill, Shilling, & Ball, 2005). They designed and tested measurement items to assess teachers' mathematical knowledge for teaching (Hill et al., 2004, 2005). They developed 138 mathematics items in two content areas, number concepts and operations, categorized them into two domains- knowledge of content and knowledge of students and content (Hill et al., 2004)- and tested these items to ensure their reliability. Based on these mathematics items, they developed their own survey instrument to collect data on first and third graders' achievement, teachers' mathematical knowledge for teaching, parents, and schools from 53 schools in 2000-2001 and an additional 62 schools in 2001-2002. The researchers found a significant effect of teachers' mathematical knowledge on student achievement (Hill et al., 2005).

Hill and her colleagues' (2004, 2005) work is groundbreaking because they used a nationally representative sample, designed reliable measurement items, and identified the effects of mathematical knowledge for teaching on student achievement. But also, their measurement items can be adapted to the studies of PCK and set in various contexts like in Ireland (see Delaney et al., 2008). Studies in China also use surveys quite frequently (Li et al, 2006; Zhao et al., 2010). For example, Zhao and his colleagues (2010) conducted survey research in six provinces, including Beijing, Shanghai, Heilongjiang, Jilin, Hebei, and Guangxi to investigate student teachers' pedagogical content knowledge of fraction and division. The findings of their research were generalized to the entire

nation, since they used a nationally representative sample with participants from north to south and from more developed to less developed areas of China.

However, the survey method is not without its problems. Critics question the validity of survey items, especially those with multiple-choice items, which have appeared in most survey designs, such as in Ball and her colleagues' studies (Hill, Rowan, & Ball, 2004; Hill, Shilling, & Ball, 2005) and Krauss and colleagues' studies (Krauss et al., 2008). The assumption underlying multiple-choice items is that there exists a right answer. However, multiple-choice items do not take into consideration context and content, which helps to define a specific teaching and learning style. This results in analyzed data that may not accurately explain the population because the survey inherently assumes that the context surrounding the original design is similar to the context of the current study. Thus, findings unique to that specific context are more likely neglected (Baxter & Lederman, 1999).

Questionnaire and interview.

Questionnaires with context-specific problems, mathematical tasks, or scenarios are always commonly used to assess teachers' PCK, as these can overcome the shortcoming of context oversimplification found in a multiple-choice item design (An et al., 2004; Turnuklu & Yesildere, 2007). For example, Turnuklu and Yesildere (2007) designed four mathematical problems to assess student teachers' interpretations of students' misconceptions or misunderstandings of mathematical knowledge. One of the mathematical problems appearing in their questionnaire is as follows:

Orcun is a 7^{th} grade student. The dialog between Orcun and his teacher is presented below.

Orcun: 5 minus 3 equal 2.

Teacher: Why do you think like this?

Orcun: I had five apples. I ate three of them. So I have two apples left.

Teacher: What is the result of -3+5?

Orcun: -3 + 5 is -8.

Teacher: How did you do it?

Orcun: 3 plus 5 is 8. The sum has the sign of the first integer.

What prerequisite knowledge might Orcun not have?

What kind of questions can be asked to Orcun to understand his misconception?

What kind of real world activity can be done to help him? (Turnuklu & Yesildere, 2007, p. 5)

It can be argued that a carefully designed questionnaire item like the one above could not only "situate" teachers in a content-specific context, but also help researchers elicit participants' thinking step by step. In addition, interview questions are often combined with questionnaire items, to help researchers elicit more in-depth responses. Thus, interview questions paired with questionnaire items serve as an even more in-depth instrument for probing PCK.

Interviews are actually the most frequently used instrument in studies of teachers' pedagogical content knowledge. Approximately more than half of the studies in this field used interview, either as the main or supplementary instrument. Interviewing can serve as a good technique to elicit teacher cognition, including teachers' pedagogical content knowledge (Kagan, 1990). Usually, interview questions are asked after each observation

in the classroom to make sure that what researchers "see" accurately represent what teachers really know (e.g., An et al., 2004).). Interview questions are also usually conducted after questionnaires to elicit teachers' thinking or reasoning underlying responses. For example, in Isiksal and Cakiroglu's (2011) study on prospective teachers' pedagogical content knowledge of multiplication of fractions, they used semi-structured interview after the Multiplication of Fraction Questionnaire (MFQ) in order to obtain a more complete picture of teachers' PCK. Here are some of their sample interview questions,

"What do you mean by…"

"Here you mentioned that…"

"Tell me more on …" (if there is something that is not clear to the researcher on the questionnaire)

"Why do you think so …" (if there is something that is not clear to the researcher on the questionnaire) (Isiksal & Cakiroglu, 2011, p. 228)

It is obvious that the questions the researchers posed were in response to prospective teachers' answers to items the questionnaires. These interview questions not only helped researchers clarify information obtained from the questionnaire, but also helped researchers probe teachers' thinking for more in-depth responses.

Reflective journal.

Using reflective journals as data is not new in studies on teachers' PCK. For example, Capraro and his colleagues (2008) asked preservice teachers in their study to write individual reflections on their teaching performances after each session throughout their senior method course in the preparation program for evaluation. Similar instruments

can be found in Richardson's (2009) research, in which participants were asked to reflect on and write about their teaching experiences in journal entries. Reflective journals are the best window into teachers' thinking. They provide rich information on how teachers view their learning, teaching practices, and teaching beliefs. However, it should be noted that when teachers are asked to submit reflections for evaluation, they may articulate perceptions that will sound "right" or "logical" to the researcher or evaluator. Thus, we may risk distorting teachers' authentic thinking.

Lesson plan.

A lesson plan is another important instrument closely related to how a teacher constructs a lesson (e.g., Cai, 2005; Cai & Wang, 2006; Ozmantar et al., 2010). Cai (2005) and Cai and Wang (2006) used teachers' introductory lesson plans on arithmetic average and on ratio and proportion to explore how teachers used various examples and representations to promote students' thinking. Ozmantar and his colleagues (2011) used pro-service math teachers' introductory lesson plans on derivatives to explore their use of multiple representations in a technologically rich environment. In addition to providing information on teachers' "objectives, prerequisite knowledge, materials used, classroom organization, outline of teacher and student activities and assessment during and after the lesson" (Ozmantar et al., 2010, p. 26), lesson plan are also useful tools for comparative studies (Cai, 2005; Huang & Cai, 2006). As Cai (2005) stated, "it was not always feasible to observe and videotape all the U.S. and Chinese teachers' teaching on the same topic" because there were "great variations between U.S. and Chinese mathematical curricula" (p. 141).

Advocacy for multi-method approaches in studies on PCK.

It is arguable that every method or instrument has its strengths and weaknesses. In pursuit of validity and reliability of data, multiple methods should be included in a single study to account for a method's or instrument's weakness. In fact, most studies on teachers' pedagogical content knowledge employ multiple methods - whether they are aligned with a qualitative tradition and/or quantitative tradition. For example, Livingston and Borko (1990) used observations, interviews and written documents to gather data on high school teachers' review lessons for their qualitative study. Shanghai Qingpu Experiment Institute (2007) combined written document, video-taping, participant observations, in-depth interviews, tests and questionnaires to study Chinese math teachers' PCK within an overall qualitative tradition. Within the quantitative tradition, Carpenter and his colleagues (1989) used observations, questionnaires and interviews to assess elementary math teachers' knowledge of their students especially of students' strategies for solving problems. Capraro and his colleagues (2008) used tests, observations, reflective journals, and interviews to explore preservice teachers' development of PCK throughout their method courses. The assessment of mathematics teachers' PCK requires a similar combination of methods (or instruments), as evidenced by the above examples, in order to improve the internal validity and reliability of data. Researchers need to triangulate data from multiple sources in order to develop a more accurate profile of teachers' pedagogical content knowledge.

Teaching Functions

The topic of functions is selected as the content area for the present study. In this section, I will review relevant studies on teaching (and learning) functions in order to

provide a content-specific context for the present study. I first briefly explain why teaching and learning functions in school is important. Then, I present a discussion on what teachers need to know when teaching functions. Finally, I provide a description of learning standards for functions in both China and the U.S.

The importance of teaching functions in school.

Mathematical "functions" has long been an important (and intriguing) topic in secondary mathematics education. A large amount of research has been done on teaching and learning functions. Why should we care about teachers' teaching functions? The necessity to teach functions comes from the importance to learn functions. The topic of functions is important for students to learn because it is the foundation for their entire mathematics education and for real-life situations. Cooney and his colleagues (Cooney, Brown, Dossey, Schrage, & Wittmann, 1996) summarize the reasons why students need to learn functions as follows: First, functions provide a context for developing basic skills such as solving equations and graphing; second, functions give students an opportunity to deal with mathematical language and representations including the symbolism of mathematics, as well as to translate among different representations such as formulas, sets, mapping, and graphs; third, functions provide students an opportunity to see how mathematics can describe real-world phenomena; fourth, functions build a foundation for the rest of the topics in secondary school mathematics; fifth, functions give students an opportunity to study functional relationships or to see how one variable changes when another variable changes. Thus, its importance in the educational system and for use in real-world situations recommends functions as an important topic to include in this study.

What do mathematics teachers need to know for teaching functions?

As mentioned above, there are two central components in Shulman's (1986) construct of PCK – teachers' content-specific knowledge of instruction and of student conceptions and misconceptions. Within teachers' content-specific knowledge of instruction are two important elements: (1) teachers' construction of mathematical tasks for student to learn functions and (2) teachers' construction of representations of functions. Based on these constructs and elements, I will present the existing literature on functions along three lines: mathematical tasks of functions, representations of functions, and teachers' knowledge of student thinking of functions.

Knowing mathematical tasks of functions.

Bell and Janvier (1981) were among the pioneers who called for developing appropriate mathematical tasks to develop students' skills in functions. In their studies, they designed several tasks to investigate students' graph reading capabilities (Bell & Janvier, 1981; Janvier, 1981; Janvier, 1985). One of their tasks asked students to interpret a pair of graphs on boys' and girls' (ages 0 to 20) height and weight increases (Bell & Janvier, 1981). This kind of task not only requires students to look at the overall structure of the graphs, or use "global" perspective to read graphs (p. 37), but it also requires them to make interpolation and comparisons between points as well as to recognize distractors (Bell & Janvier, 1981). The careful design of tasks is crucial for student learning. Even tasks requiring the same mathematical knowledge may have very different levels of difficulty that emphasizes different cognitive processes (Hazzan & Zazkis, 1999). For instance, "standard execution" tasks such as calculating the value of a function at a given

point is much easier than constructing an example/object with given properties (Hazzan & Zazkis, 1999, p. 11).

Most tasks closely related to functions and graphs can be categorized into prediction tasks, classification tasks, translation tasks, and scaling tasks, especially in the lower and middle levels of functions learning (Leinhardt, Zasklavsky & Stein, 1990). Prediction involves conjectures from a given part of an equation or a graph, as well as detection of patterns. Classification usually involves decisions on whether a particular relation is a function as well as understandings of students' concept images of a function (Vinner & Dreyfus, 1989; Leinhardt et al., 1990). Translation refers to recognizing the same function in different forms of representations and constructing one presentation of a function based on another. Scaling involves decisions on scales and units that are specific to the domain of graph.

Additionally, based on what action students need to take, mathematical tasks can be examined along an interpretative or a constructive dimension (Leinhardt et al., 1990). If a teacher asks his/her students to describe a particular point in a graph, the teacher constructed an interpretive task because it needs students to "make sense or gain meanings from a graph (or a portion of a graph), a functional equation, or a situation" (Leinhardt et al., 1990, p. 8). Or, if a teacher asks a student to create an equation from a graph, the teacher developed a constructive task because it needs the student to "generate something new" or to "build a graph or plot points from data (or from a function rule or a table) or to build an algebraic function for a graph" (Leinhardt et al., 1990, p. 12). Prediction tasks usually require constructive actions (Leinhardt et al., 1990). Research so far, however, has paid more attention to interpretative features of tasks, while

constructive tasks remain have been ignored (Bell & Janvier, 1981; Dreyfus & Eisenberg, 1982; Vinner, 1983; Leinhardt et al., 1990).

Other themes from studies on mathematical tasks of functions include (but not exhaustively): building connections between concepts; connecting to real world; and promoting advanced abstract thinking. Hazzan (Hazzan, 1996; Hazzan & Zazkis, 1999) used the "give an example" task to show how this kind of mathematical task helped students connect concepts. In this case, students were asked to give a function with the value -2 at x=3 (Hazzan & Zazkis, 1999). They pointed out that, when students were asked to give an example of functions, they actually constructed a link connecting "the concept of function and a function-value-at-a-certain-point" (Hazzan & Zazkis, 1999, p.4). Some authentic mathematical tasks are used by researchers to provide students with a real-world context to learn mathematics. In McGraw, Romero and Krueger's (2006) stadium seating problem, students were asked to determine the relationship between the bleacher number (x) and the height of the bleacher above the ground (y) by using tables, diagrams and equations. The task was posed progressively from a point-wise question – determining a point through reading a graph - to a global question – determining the equation. In this way, they not only helped students investigate a linear function but also make connections across multiple representations. Another theme of studying mathematical tasks of functions is using mathematical modeling tasks, which aims to advance students' abstract thinking (Yoon, Dreyfus, & Thomas, 2010). Yoon, Dreyfus, and Thomas (2010) investigated two undergraduate students' and two secondary math teachers' performances on finding a representation of an anti-derivative of a function. Participants found it easier to apply what they learned in the real world. However, when

it came to modeling, they found it much more difficult since modeling involved connecting the real world and the mathematical world.

Knowing representations of functions.

Janvier (1985) understood representations from three psychological aspects. The first is that a representation is the material organization of symbols such as diagrams, graphs, schemas, etc. He sometimes called it schematization or illustration. The second aspect refers to the organization of knowledge in the human mental "system" or "in the long-term memory" (Janvier, 1985, p. 5). Sometimes we use "conception" to refer to this meaning. The third refers to mental images, which is a "special case" of the second aspect.

Another way to categorize the system of representation is to use the classifications of "external representation" and "internal representations" (Goldin & Shteingold, 2001). External representations include the conventional symbol systems of mathematics, as well as structured learning environments (Goldin & Shteingold, 2001). In other words, external representations can be understood as any visible signs, characters, and/or objects that stand for something other than itself. This is similar to Janvier's (1985) first meaning of representation. Internal representations refer to how students personally construct symbols, allocate meaning to mathematical notions, and select what strategies to use to solve a problem. This is similar to Janvier's (1985) second and third meaning of representation. The focus of this study is external (and therefore observable or seeable) representations.

It is agreed that it is important for teachers to select pedagogically sound representations for teaching (Ball, 1993; Leinhardt, 2001). Pedagogical representations

related to the teaching of functions are usually classified into five categories: verbal, tabular, numerical, symbolic and graphical representations (Cuoco, 2001). Verbal representations are sometimes also called literal representations, which refer to some forms of description of functions. They are usually used in word problems and final interpretations of solved problems. Tabular representations refer to functions or equations in the form of tables. Numerical representations refer to functions or equations represented by ordered pairs, which are often in the initial stage of functions learning to help students obtain a first-hand understanding of a problem. Symbolic representations are sometimes called algebraic representations, which refer to functions or equations that utilize two variables. This type of representation is "concise, general, and effective in the presentation of patterns and mathematical models" (Friedlander & Tabach, 2001, p. 174). Graphical representations refer to functions shown on a coordinate plane, which can help to provide a clear and tangible depiction of functions and variables.

Teaching functions is difficult partly because functions involve a variety of representations (Janvier, 1985) as well as the translations among different representations of the concept of function. For example, teachers were encouraged to integrate graphs into the learning of functions in class (Norman, 1993). In order to engage students in learning functions, teachers should have a solid foundation of representations of functions. That is, they should be able to use multiple representations to promote students' conceptual understanding as well as procedural understanding of the concept of function. Based on research with 162 college students' responses to a series of function problems, Even (1998) found that different representations of functions were interconnected or "intertwined" in function problem solving. In other words, knowledge of different ways

to approach functions, such as global and point-wise approaches, recognition of the context of the function problem, and identification of the underlying notions of the problem in relation to different forms of representations contributed to this interconnection. A global approach was usually more powerful than a point-wise approach, especially when translating between representations (e.g., graphic and symbolic representations), but the point-wise approach "was helpful in monitoring naïve and/or immature interpretations" (Even, 1998, p. 120). He also supported the idea that "a combination of the two methods is most powerful" (p. 120), which was supported in a later research by Izsak and Sherin (2003). Based on their investigation of teachers' use of representations when teaching linear functions, they argued that making connections between algebraic and graphical representations of functions was critical for a deep understanding of functions. One aid that makes the use of multiple representations of functions more convenient and appealing to teach is technology (e.g., Ozmantar et al., 2010). Ozmantar and his colleagues (2010) discussed how the use of software can help teachers generate graphical, tabular, and numerical representations for teaching slope and derivative functions. Translation among different representations is also crucial to understanding the concept of function. Lagrange's (2010) research showed how teachers can use multiple representations to teach 11^{th} grade functions. The researcher argued that a project taking advantage of the interconnectedness of representations can help students grasp the "conversion of the graphic to the symbolic register" (Lagrange, 2010, p. 253) of functions.

Although the use of multiple representations is highly encouraged, the reality is that teachers do not often utilize multiple representations (Friedlander & Tabach, 2001).

Rather, they often select one or two representations they are familiar with in class (Even, 1993). This can hinder their students from building a complete picture of representations (Even, 1993; Huang & Cai, 2011). Thus, it is important for teachers to utilize multiple representations. Additionally, they should use representations that are consistent with student needs and context of the function (Meyer, 2001; Yerushalmy & Shternberg, 2001). For example, in order to help students in lower grades deal with graph problems in functions, the combination of graphs and tables would be the most effective approach because graphs and tables are more intuitive and tangible for children in lower grades (Bell & Janvier, 1981; Moore-Rossuo & Golzy, 2005).

Knowing students' thinking of functions.

How do students construct the concept of function in their minds?

To understand students' conceptions and misconceptions of functions, teachers first need to know how students construct this concept in their minds. Vinner and his colleagues have conducted a series of important studies on students' "concept images" and "concept definitions" of the concept of function (Tall & Vinner, 1981, p. 151. See also Vinner, 1976, 1983; Vinner & Dreyfus, 1989). A concept image is a mental picture, such as symbols and graphs, with its properties that students conceive in their minds (Tall & Vinner, 1981, p. 152). Concept definition refers to a verbal definition, which can "accurately explain the concept in a non-circular way" (Vinner, 1983, p. 293). Discrepancies are found between students' concept image and definition of functions. That is, students' formal definitions of a function are not consistent with their image of the function concept (Tall & Vinner, 1981; Vinner & Dreyfus, 1989). An introduction (or indoctrination) of concept definitions and some examples do not necessarily lead to

complete and correct concept images in students' minds (Vinner, 1983; Vinner & Dreyfus, 1989). Thus, it is important for teachers to understand how students form and use concept images when they teach. Efforts have been made to understand students' conception of functions. For example, Dubinsky and Harel (1992) identified four factors related to students' ways of thinking about functions based on observations and interviews with 22 students in a Discrete Mathematics Class. These factors are: (1) restrictions that students possess about what a function is; (2) the severity of the restriction; (3) the ability to construct a process when none is explicit in the situation; and (4) uniqueness to the right conditions.

What difficulties do students usually encounter in learning functions?

Teachers need to know what difficulties their students usually encounter when learning functions. Researchers have been dedicated to answering this question over the past few decades. Different sources of difficulties, though not exhaustive, have been identified. The first source of difficulty lies in the "intrinsic ambiguities" in the mathematical notation of functions (Sajka, 2003). This makes the comprehension of the concept of function extraordinarily complex (Selden & Selden, 1992). Halmos (1974) defined a function as,

> If X and Y are sets, a function from (or on) X to (or into) Y is a relation f such as that dom $f = X$ and such that for each x in X there is unique element y in Y with $(x, y) \in f$. (p. 30)

Sfard (1991, 1992) points out that the concept of function can be understood in dual ways: a structural way with the functions being defined as a set of ordered pairs (see Kuratowski & Mostowski, 1966); and an operational way with the functions being

defined as “a computational process” (see Skemp, 1971). Students are more likely to choose the operational version of the definition of functions. In Sfard’s (1992) research, 81% of the students chose the description that associated functions with computational processes, while only 19% of the students viewed functions from a structural perspective.

The second source of difficulty in learning functions lies in the “context” of student learning (Cooney et al., 1996; Sajka, 2003). Students only learn functions and the concepts related to functions in the school setting, or more exactly, in the classroom. They learn symbols, formulas, and graphs, but they are not trained to use this knowledge and skills in real life. They are rarely given an opportunity to recognize that functions are actually used in the media and in everyday conversations.

The third source of difficulty is what Sierpinska (1992) calls “epistemological obstacle”. The difficulties encountered by students, especially middle and high school students, are attributed to the understanding that functions are related to the “philosophy of mathematics and mathematical methods and various unconscious schemes of thinking” (Sierpinska, 1992). In other words, students are used to conceiving mathematics from a particular approach, which is why they may have difficulty when they are expected to use another approach. For example, students usually find it difficult to shift from distinguishing between known and unknown quantities – which they have experienced in previous mathematics learning – to distinguishing between constants and the variables, since it involves a shift to a different (and higher level) mode of thinking (Sierpinska, 1992).

The fourth difficulty that students usually encounter in functions is the idea of thinking globally (McDermott, 1987; Monk, 1992). In Monk’s (1992) study, students

were given a graph of speed vs. time for two cars. It was easier for students to give correct answers to the speeds of the two cars at a "particular time", but difficult for them to give correct answers when asked to "describe the relative positions of the two cars at time t=1hr" (Monk, 1992, p. 175). Monk (1992) pointed out that it was because "Across-Time" questions required students to determine the patterns of change in the values of the input variables" (p. 176), while point-wise questions only asked for a specific value of a function.

All these obstacles, which hinder students' learning of functions, pose a challenge to math teachers.

What prior knowledge do students have before studying functions?

Prior knowledge plays an important role in students' understandings of specific topics (Grossman, 1989; NCTM, 2000), especially when they are exposed to a new concept. Students build upon their foundations of mathematics by connecting topics they learned. As NCTM (2000) stated, "Because students learn by connecting new ideas to prior knowledge, teachers must understand what their students already know" (p. 18). To understand functions, Sierpinska (1992) pointed out that "a certain awareness of algebraic methods and of algebra as a methodological tool in mathematics is necessary for studying functions (p. 44)". Thus, before the general concept of function is introduced to students, they should be exposed to algebra, such as using letters for unknown quantities and solving simple linear equations (Sierpinska, 1992). This prior knowledge is quite important for student learn functions. It is sometimes quite effective to also revisit a prior problem to help students then connect familiar ideas to any new concepts or skills. Vinner and Dreyfus (1989) pointed out the importance of teachers' abilities to connect

the concept of function for their students. For instance, the examples that teachers bring to class should be an extension of students' previous experiences with functions. Hazzan and Zazkis (1999) found that students' abilities to construct conceptual links among mathematical concepts is relatively weak based on three author-constructed tasks in which students were asked to "give an example" of divisible numbers, functions, and equations with their respective properties. Thus, teachers need to consistently help students develop the ability of making those connections.

What misconceptions do students usually have about functions?

Teachers should also be aware of students' misconceptions of mathematics and be able to address such misconceptions (An et al., 2004; Vinner, 1983). A variety of misconceptions have been identified by researchers. Some of the typical misconceptions are listed as follows. First, students tend to equate computational formulas to functions (Even, 1993; Sfard, 1992; Sierpinska, 1992; etc.). For example, Sfard (1992) found that students were more likely to understand functions in an "operational" way and further develop "pseudostructural conceptions" of functions (p. 75). That is, they tended to describe a function as a "certain computational formula" (Sfard, 1992, p. 75). A second typical misconception is that constant functions and functions with split domains are not actually functions (Even, 1993). Third, a function is its representation (Sfard, 1992). For example, some students tend to describe a function as an equation. Fourth, students tend to believe that all functions are continuous (Carlson, 1998). Different approaches of addressing students' misconceptions are identified in some of the research. Teachers use models and activities and connect these models and activities to abstract thinking to help

students correct their misconceptions. They also use probing questions to identify students' mistakes or errors (An et al., 2004)

What do standards in China and the U.S. say about functions?

Both China and the U.S. have been dedicated to implementing standards-based education reform in the past decade. Standards in both countries serve as guides for what students should know and be able to do at any given grade. In this section, I review standards in both countries within the context of functions.

In 2011, China issued and implemented its new national mathematics standards (*New Standards*) for compulsory education (from the 1st through 9th grade). According to the *New Standards* (Chinese Department of Education, 2011), the concept of function is formally introduced to students at the 8th Grade level. The main goals of the *New Standards* are for students to (1) understand the concept of function and its representations; (2) be able to make distinctions between constants and variables; (3) be able to give examples of linear and nonlinear functions; (4) understand linear functions and being able to construct (and interpret) functions through reading graphs and vice versa; and (5) understanding quadratic functions.

In the U.S., the National Council of Teachers of Mathematics (NCTM) published its mathematics standards for schools in the nationwide, though its implementation is not mandatory. The NCTM's (2000) Algebra Standard of functions at the middle school level (6th - 8th grades) states that students should: solve problems in which they use tables, graphs, words, and symbolic expressions to represent and examine functions and patterns of change (p. 223); and be able to identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations (p. 222). State governments then publish

their own mathematics standards, usually incorporating what is recommended in the national standards. In Arizona, the *Arizona Academic Content Standards: Mathematics* was approved by the Arizona State Board of Education in 2010 and implemented statewide thereafter. *Arizona Academic Content Standards: Mathematics* (Arizona Department of Education, 2010) specifies goals that students in middle school are expected to achieve. These include: (1) understanding that a function is a rule that assigns to each input exactly one output; (2) comparing properties of two functions, where each is represented in a different way; (3) being able to construct a function to model a linear relationship between two quantities; (4) being able to give examples of functions that are not linear; and (5) being able to describe qualitatively the functional relationship between two quantities on a graph.

It is noted that students, both in China and the U.S., are formally introduced to the concept of function at the 8^{th} grade level. Both countries share key elements in what they aim to achieve with their students. These are: (1) understanding the concept of function; (2) understanding patterns that relate to linear functions; (3) using multiple representations of functions, such as symbolic and verbal, and moving flexibly among these representations; (4) analyzing the effects of parameters/variables; and (5) analyzing change in various contexts (Arizona Department of Education, 2010; Chinese Department of Education, 2011; NCTM, 2000). There is one main difference regarding the coverage of content. In China, the middle school mathematics standards stipulate that students should understand quadratic functions. However, this is not the case for the middle school standards in the U.S. Nonetheless, students in both countries don't actually learn quadratic functions until the 9^{th} grade.

Concluding Remarks

The literature review I have developed here provides an overview of the research on mathematics teachers' pedagogical content knowledge. Specifically, I presented a general overview of PCK research from a theoretical, empirical, and methodological perspective and then contextualized these studies within the frame of functions.

Although it has been two decades since Shulman (1986, 1987) posited the term of PCK, the theoretical perspectives of PCK remain scattered and the domains and measures of PCK are underspecified. More research should be devoted to testing the theoretical hypotheses and specifying the domains and measures of PCK.

Empirically, although a variety of research has been conducted to reveal the state of mathematics teachers' PCK, a number of dimensions have been overlooked. First, experienced math teachers and their current state of PCK are rarely examined and discussed compared to novice (and prospective) math teachers. Second, although cross-national comparisons of teachers' PCK have been flourishing, China-U.S. comparisons are scarce in the literature and most of them are focused on elementary math teachers. Third, although the findings regarding comparisons of elementary mathematics teachers are generally positive toward Chinese teachers, the comparisons regarding middle school mathematics teachers are unknown. More comparisons on China and U.S. middle school mathematics teachers are needed.

I also reviewed instruments to measure PCK as an external and an internal construct. Although class observations, associated with the external construct of PCK, are used in many studies, there is much or greater validity in using instruments that measure the internal construct of PCK (Kagan, 1990). Additionally, research using a variety of

instruments to measure mathematics teachers' PCK indicates that using multiple methods in a study is a better way to overcome the shortcomings of any single instrument and to triangle also data and findings.

Lastly, an additional literature review was done on mathematics teachers' PCK in the context of functions. I selected functions because of its importance in the academic and real-world settings. Additionally, functions will be used as the context of investigating mathematics teachers' PCK in this study. Based on the gaps in research, this study investigates Chinese and U.S. middle school mathematics teachers' pedagogical content knowledge of functions.

The research questions are:

1. What instructional decisions do the Chinese and U.S. middle school mathematics teachers make to introduce the concept of function?

What instructional goals do Chinese and U.S. math teachers set for class? How do they explain these goals?

What mathematical tasks do Chinese and U.S. math teachers construct? How do they explain their construction of these tasks?

What representations do the teachers use? How do they explain their use of representations?

2. How do the Chinese and U.S. middle school mathematics teachers deal with students' mistakes in functions?

What math ideas do they suggest that students need to know in order to correctly solve the math problems?

What knowledge do they have about student thinking when solving the problems

in functions?

What strategies do they use to correct the misconceptions?

3. What knowledge do Chinese and U.S. middle school mathematics teachers have about the mathematics curriculum?

How do they use instructional materials to teach the topic of functions?

How do they understand functions as related to other subject areas?

What knowledge do they have regarding functions as a topic in previous and future grades?

In the next chapter, I present a detailed description of my research design used to answer these questions.

CHAPTER 3

METHODOLOGY

The goal of the present study is to produce a descriptive piece of research that captures the state of Chinese and U.S. middle school mathematics teachers' pedagogical content knowledge regarding functions. To achieve the goal of the present study, I consider this investigation a comparative, descriptive case study within the qualitative methodological tradition.

As Pickard (2007) stated, the choice between quantitative and qualitative was the highest level of methodological decisions. I conduct this study within the qualitative tradition, as qualitative method is "committed to the naturalistic perspective, and to the interpretive understanding of human experience." (Denzin & Lincoln, 1994, p. 3) For my study, the qualitative methodology is more likely to help craft a dense text on the current state of teachers' pedagogical content knowledge in both countries.

This study is considered to be a comparative, descriptive case study (Merriam, 1998; Stake, 1995, 2003; Yin, 1993, 2003). First of all, this study is a case study because it (1) focuses on a particular phenomenon (Merriam, 1998; Yin, 2003) - middle school mathematics teachers' PCK, (2) emphasizes an extensive dialogue between the researcher's ideas and the data (Yin, 1993, 2003), and (3) needs the researcher to spend extended time in contact with the teachers- in order to collect rich data on teachers' PCK – and to "reflect and revise meaning of what is going on" (Stake, 2003, p. 203). Second, this study is a descriptive case study because the main objective of this study is to present a "thick description" (Merriam, 1998) of teachers' PCK. This "thick description" covers the scope and depth of the teachers' PCK in both countries and it is not aimed at

producing a cause-effect relationship (Yin, 1993, 2003). Third, a comparison is made to identify similarities and differences of teachers' pedagogical content knowledge between China and the U.S. The unit of analysis is individual middle school math teacher. Each country can be viewed as "a context" in which multiple cases are located. These cases are individual middle school math teachers' PCK. Multiple cases in these two contexts are then contrasted to present a more in-depth picture of the similarities and differences between teachers' PCK in China and the U.S. Thus, it can also be regarded as an extension of "collective case studies" (Stake, 1995) or the "multiple-case studies" (Yin, 1993, 2003) to the cross-national context.

Selection of Participants

In this study, participants- middle school mathematics teachers- were "purposefully selected" from the public school systems in each country in order to select "information-rich cases for study in depth" (Patton, 2002, p. 169). Specifically, I used "a priori sampling approach" (Flick, 2002), which identified criteria and established a sample framework before sampling began, to purposefully select my participants. The criteria and the procedures of selecting participants are described as follows.

Criteria of the selection of participants.

Middle school mathematics teachers for this study were recruited from schools in the Phoenix metropolitan area (in the U.S.) and the Beijing metropolitan area (in China). The Phoenix metropolitan area and Beijing metropolitan area were selected primarily because of their shared high density populations including the number of school-age children, and the complexity of education in each urban area. Charter schools and online learning institutions in these two sites were excluded from this study because there were

no charter schools in China on the one hand, and online learning institutions took a different form of teaching on the other.

I aimed to select teachers both from schools with students of high SES social-economic status (SES) and from schools with students of low SES in each area. The relationship between the school performance or school quality and the socio-economic status of student populations within schools is complex. They are, however, intertwined. As Hill and her colleagues (Hill et al., 2005; Hill, 2007; Hill & Lubienski, 2007) stated, schools with poor populations were consistently linked to lower school performances and had lower teacher quality, while schools with affluent populations were always linked to higher school performance and had higher teacher quality. It was not surprising to see high performing schools usually have students with higher SES in Beijing and Phoenix. In the Phoenix metropolitan area, the major economic indicator used to include potential schools from which the participants were selected was the percentage of students having free/reduced lunch. The average percentage of students having free/reduced lunch in the state of Arizona was 57.06% when the data collection began (Arizona Department of Education, 2012), which served as the bench mark for low SES and high SES status within schools in the Phoenix metropolitan area. In the Beijing metropolitan area, the economic indicator of selecting schools for both high and low socio-economic levels in Beijing was not the same as that in Phoenix metropolitan area, since statistical indicators were not the same in the two countries. The idea of a percentage of students having free/reduced lunch was not applicable in China. Thus, two economic indicators were taken into account to categorize and select schools in Beijing: the house prices in the neighborhoods in which each school was located and the average family income of the

student population within each school. The house prices in neighboring areas and the average family income of the student population within a school imply the SES of the student population and further imply the quality of the schools (Feng & Lu, 2010; Zheng & Kahn, 2007).

In order to examine teachers' pedagogical content knowledge in the area of functions at the middle school level, this study was aimed to recruit U.S. teachers from 7^{th} through 9^{th} grades and Chinese teachers from 7^{th} through 9^{th} grades. These grade levels were chosen for China because middle school in China usually includes grades 7, 8 and 9 and this organization is quite uniform within the country. Functions are formally introduced as a new mathematical concept at the eighth grade level. In the U.S., the term of middle school is muddies by the synonymous term, "junior high school," which signifies different grade levels than middle school. Middle school in the Phoenix metropolitan area usually includes 7^{th} and 8^{th} grades, while junior high school includes 7^{th} through 9^{th} grades. To account for this difference, I aimed to recruit teachers in the U.S. from 7^{th} through 9^{th} grades, covering middle school and junior high school. Functions are formally introduced as a new mathematical concept at the seventh grade (*old standards*) or at the eighth grade (*the Common Core Standards*).

In this study, the target participants were experienced teachers currently teaching middle school math. An experienced teacher was defined as someone who had at least five years of teaching experience (Berliner, 1988). This five-year mark was used as a cutoff point from which to filter out novice teachers. Teachers' certificates were examined to make the samples in both countries more comparable. In China, especially in Beijing, most mathematics teachers in middle schools majored in mathematics at normal

colleges/universities and were then trained to teach at the middle school or secondary school (Chinese Department of Education, 2010). Thus, in terms of China, I aimed to recruit teachers who had at least middle school math certificate. The situation of the U.S. was more complex regarding teachers' backgrounds. In Arizona, teachers had been required to pass the subject matter exam and knowledge of teaching exam to get teaching certificate(s) by the year of 2011 (NCES, 2011). However, there was overlap between the different levels of teaching certificates such that the elementary certificate, which authorized teachers to teach from K-8, overlapped with the secondary certificate, which authorized teachers to teach from grades 6-12. Gaps were found in mathematical knowledge for teaching between middle school teachers credentialed for grades 6 -12 and those credentialed for grades K-8 (Hill, 2007). In Arizona, the state allowed middle school teachers to teach on a K-8 generalist license but it also offered an optional middle grades endorsement (grade 5-9) for teachers "to expand the grades a teacher is authorized to teach on an elementary or secondary certificate" (National Council on Teacher Quality (NCTQ), 2009, p. 25). Therefore, to make the US sample comparable to the Chinese sample, I chose to include those with this secondary certificate in mathematics, middle school certificate in mathematics, and those with elementary certificates, but having concentration (or extension) in middle school mathematics.

Procedures of the recruitment of participants.

Based on the criteria stated in the last section, eligible mathematics teachers constituted the sampling pool for this study. In the Phoenix metropolitan area, letters were directly sent via email to those middle school math teachers who met all the criteria, inviting them to participate in the present study. Teachers' detailed information on school

websites helped in the screening of potential participants in the Phoenix metropolitan area. Invitations were emailed to over 300 teachers from over 90 schools located in about 30 school districts. 31 teachers responded that they were willing to participate in my study. The response rate was about 10%. In the Beijing metropolitan area, I used two approaches to recruit participants. The first approach was similar to that in the Phoenix metropolitan area. Only nine teachers were willing to participant in my study using this approach. The response rate was about 3%, much lower than that in Phoenix, however, that may have been due partly to cultural differences. Thus, I chose to use a second approach - asking my colleagues in Beijing to invite teachers to participate in my study through their social networks. These colleagues (insiders) also helped identify the SES status of each school. 25 teachers responded that they were willing to participate. Those who chose to participate were asked to sign a participant form before participation.

Participants of this study.

There were initially 65 teachers who were willing to participate in this study. Fourteen of them withdrew before the data collection began. Thus, the final sample of participants in this study included 51 experienced math teachers who currently taught middle school math in public schools in the U.S. and China. 23 teachers in Arizona and 28 teachers in Beijing participated in this study (see detailed in Table 1). Specifically, the U.S. lower SES teacher group included twelve middle school math teachers who came from nine schools located in nine school districts. Their teaching experience ranged from five to sixteen years, with an average of 10.4 years. Two of them were males, while the other ten were females. Seven of them held a secondary math certificate, four of them held a middle school math certificate, and one of them had a K-8 certificate with a

concentration on middle school math. The U.S. higher SES teacher group included eleven middle school math teachers who came from nine schools located in seven school districts. Their teaching experience ranged from six to twenty-six years, with an average of 14.8 years. One of them was male, while the other ten were females. Seven of them held a secondary math certificate, two of them held a middle school math certificate, and two of them had a K-8 certificate with a concentration on middle school math. In China, the lower SES middle school teacher group included fourteen participants, with four males and ten females. These teachers came from seven schools that were located in three school districts (note: school districts in Beijing are much larger than school districts in Phoenix). The average years of teaching were 14.2. Twelve teachers held secondary math certificates while the other two held middle school math certificates. The Chinese higher SES middle school teacher group also included fourteen participants, with five males and nine females. These teachers came from eight schools that were located in four school districts. The average years of teaching were 12.6. Nine teachers held secondary math certificates and the other five held middle school math certificates.

Table 1

Sample of Participants

	U.S. lower SES	U.S. higher SES	China lower SES	China higher SES
Number of participants	12	11	14	14
Gender	Male- 2 Female- 10	Male- 1 Female- 10	Male- 4 Female-10	Male-5 Female-9
Average years of teaching	10.4	14.8	14.2	12.6

Certificate	Secondary- 7 Middle school-4 K-8 with concentration on middle school math -1	Secondary- 7 Middle school-2 K-8 with concentration on middle school math -2	Secondary-12 Middle school-2	Secondary-9 Middle school-5
Number of schools	9	9	7	8
Number of school districts	9	7	3	4

Participant compensation

Participants in this study devoted time for the questionnaire responding, lesson plan writing, and interviewing. Their participation in this study was compensated for in two ways: on the one hand, participants were able to obtain an opportunity to examine and analyze their own state of pedagogical content knowledge throughout the participation (Spradley, 1979). This was a spiritual compensation for their participation in this study. On the other hand, participants in the U.S. got a $25 gift card and participants in China got an equivalent gift for their participation in the study, which was a material compensation for their participation in this study.

Data Collection

The data sources of the present study included questionnaires, lesson plans and interviews, on which I elaborate in this section. Prior to data collection, a Human Subjects Application was submitted to and approved by the Institutional Review Board (IRB) of Arizona State University.

Questionnaire.

Questionnaires were helpful in gathering information from participants through questions or other prompts (Foddy, 1993; Oppenheim, 1992). Although questionnaires are not among the most prominently used methods used in qualitative research, it was useful in eliciting information on participants' ideas and conceptions. In studies on PCK, a large number of questionnaire items were constructed using scenarios or problems (e.g., An et al., 2004; Turnuklu & Yesildere, 2007) in order to understand teachers' knowledge.

In this study, a questionnaire was constructed to obtain background information about the participants and to elicit information on Chinese and U.S. middle school mathematics teachers' understandings of students' thinking. The questionnaire consisted of two sections. The first section asked background questions, such as teachers' years of teaching and certificate. The second section was the core of the questionnaire, which included two problems with open-ended questions that were designed to examine teachers' understanding of students' prior knowledge, their ability to identify students' misconceptions, and their use of strategies when correcting students' misconceptions of functions. The first scenario is adapted from Even's (1998) research (see Figure 2). The second scenario is adapted from Monk's (1992) research (See Figure 3). The questionnaires were prepared first in English, and then translated into Chinese, with a back-translation procedure to ensure its correctness and accuracy. The question items were designed drawn from other studies of teachers' PCK about mathematical functions, and modified to suit this study (An et al., 2004; Even, 1993, 1998; Huang & Cai, 2011; Javier, 1981, 1985; McDermott, Rosenquist, & vanZee, 1987). The questionnaires (both in English and Chinese) are found in Appendix C. The first problem focused on the

definition of function, and the second problem is based on the comparison of linear graphs. As mentioned above, both problems tried to address teachers' understanding of students' prior knowledge, their ability to identify students' misconceptions, and their use of strategies when correcting students' misconceptions.

Scenario 1: If a student is asked to give an example of a graph of a function that passes through the points A and B She gives an example as shown in Figure 2. When asked if there is another answer for this question, she says "No".



Figure 2 Scenario 1 of the questionnaire (Even, 1993)

A student is given the position vs. time graph as presented below. When asked to compare the speeds of the objects at time $t = 2$ sec., the student responds by saying that Object B is moving faster.

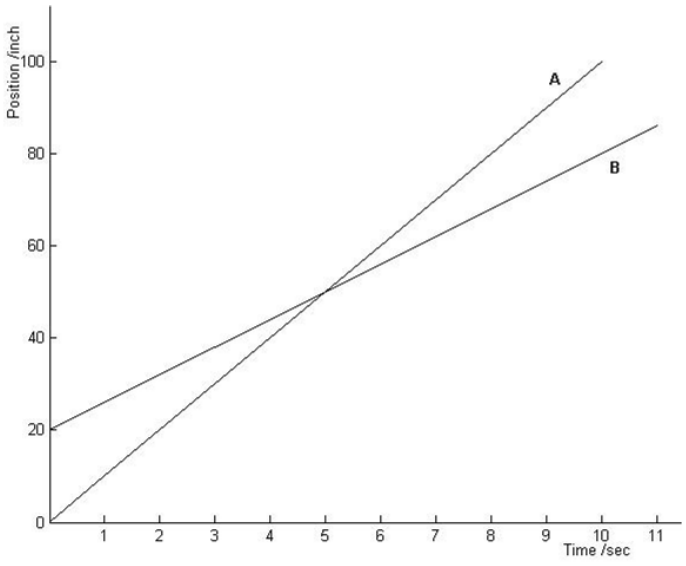


Figure 3 Scenario 2 of the questionnaire (Monk, 1992)

The questionnaire was sent out to participants via email, attached as a PDF file. Participants were asked to answer those questions and return their responses via email within two weeks. This was actually what Oppenheim (1992) called a "self-administered" questionnaire. In this process, the respondent was left alone to complete the questionnaire, which may be sent in or collected later. One concern about using a self-administered questionnaire was that participants may check document, such as a teacher handbook and website sources, or consult other people (Oppenheim, 1992), rather than answer authentically. However, it was not so threatening considering the following: first, there were no direct answers for those questions which teachers can simply "copy and paste"; second, open-ended questions required teachers to think and formulate their own answers rather than perfunctorily pick an answer from given options; and third, in reality, preparing for teaching often involved using materials and consulting other teachers. Moreover, as Oppenheim (1992, p. 34) contended, "the fact [that] no interviewer is present means that there will be no interviewer bias." A self-administered questionnaire

can help minimize interviewer bias. In sum, it appeared to be reasonable to collect questionnaire responses via email.

Time lines for acquisition.

In January, I started sending questionnaires to the participant teachers in the Phoenix metropolitan area. The 23 participants in this study were those who completed and turned in their questionnaires. This data collection was completed by mid-February in the U.S. In China, the same data collection was completed by the end of April. The 28 participants in this study were those who completed and turned in their questionnaires.

Lesson plan.

Introductory lesson plans on functions were used as the main data source. These were considered a valuable tool through which to explore what instructional decisions teachers make to introduce the concept of function. Participants in both countries were asked to write a detailed lesson plan on introducing the concept of function. I did not give the participants any template to follow. Instead, I asked them to use the templates they usually used in their teaching to write this lesson plan. However, I pointed out the content I need them to include. The lesson plan request letters are provided in Appendix A. The rationale for using lesson plans as data collection instrument was threefold.

First of all, a lesson plan is a teacher's detailed description of instruction for a particular class. Designing lesson plans provided opportunities for teachers to think about how they might teach a specific topic (Cavey & Berenson, 2005). In writing their lesson plans, teachers need to consider the goals for the particular topic, which examples they will use, which representations they will construct, which mathematical tasks and activities they will create and in what sequence, what assignments they will give to

students, etc. From this perspective, teachers are making instructional decisions and "rehearsing" them in their heads when writing these lesson plans. Thus, analyses of lesson plans can provide insights into teachers' mathematical knowledge of instruction.

Second, though lesson plans take a variety of forms, the key elements of a lesson plan are similar within and across China and the U.S. In China, a lesson plan is called "*jiao xue ji hua*" or "*jiao an*", which means a plan for teaching. Lesson plans in China and the U.S. both emphasize (1) goals and objectives of the lesson which point out what students will be able to do after the lesson and during the lesson, respectively; (2) descriptions of the lesson- providing an overview of the topic focus, activities, and purpose; (3) lesson procedures- describing the flow of the lesson, such as the sequence of activities and examples and representations that will be used; and (4) assessment, used to evaluate if the objectives for the lesson are achieved (Arizona Department of Education, 2012; U.S. Department of Education, 2003; Bu, 2012). These similar key elements of lesson plans, to a large degree, ensure comparability of the lesson plans across both countries.

Lastly, lesson plans on introducing a new concept usually require teachers to use various examples, representations, and activities, as well as to connect the new concept to related concepts (Cai, 2005). Thus, introductory lesson plans on the concept of function were a good source to get data for the first research question of the present study.

Time lines for acquisition.

In both countries participants who turned in their completed questionnaires were asked to write a full introductory lesson plan on the concept of function. I completed collecting the U.S. participants' lesson plans by early-March. Ten complete lesson plans

were collected from the U.S. lower SES middle school teacher group and ten were collected from the U.S. higher SES teacher group. In China, I completed collecting the Chinese mathematics teachers' introductory lesson plans by the end of April. Fourteen complete lesson plans were collected from the lower SES middle school teacher group and eleven were collected from the higher SES middle school teacher group.

Interviews.

Interviews are usually created to elicit in-depth participant response (Glesne, 1999; Kvale, 1996) and are also particularly useful to gain insights on how interviewees interpret "some piece of the world" (Bogdan & Biklen, 2007). As I mentioned in the previous chapter, interviews were the most frequently used tool in PCK studies because of their ability to capture ample data for understanding teachers' knowledge (Kagan, 1990). Additionally, interviews can provide opportunities to "learn about what you cannot see and to explore alternative explanations of what you do see (Glesne, 1999. P. 69)". For this study, interviews not only helped me to "see" what I cannot see from teachers' lesson plans and questionnaires, but also provide opportunity to find any "unexpected turns" (Glesne, 1999) from teachers' response.

Individual interviews, rather than focus groups, were used in this study because it allowed me to probe for more details and ensure that participants were interpreting questions the way they were intended. It also helped me to gain a lot of data in a generally simple format (Hays & Singh, 2012). I interviewed to understand how these teachers taught functions, such as their reasons behind each instructional decision and their strategies of dealing with student mistakes. Semi-structured interviewing, as a more

formal, orderly process, was used to help to collect data on teachers' knowledge about teaching functions.

"Convenient, available, appropriate" (Glesne, 1999) locations and times were chosen by both the researcher and the participants to conduct my interviews. For interview locations, quiet, physically comfortable, and familiar places were the ideal locations for conducting interviews with teachers. The locations in which my interviews were conducted included teachers' offices and classrooms in their schools (the main locations) and coffee shops near teachers' schools or near their homes. For interview times, I asked each teacher to suggest a time that was most convenient for him or her. In the U.S., these times included mainly during lunch and after school, whereas in China, these times were mainly the afternoons in school.

Participants who turned in both their completed questionnaire responses and lesson plans constituted the interviewing pool of this study. I emailed (or called) these potential participants, inviting them to take my interview. In the end, nineteen teachers were interviewed in this study. Specifically, in the U.S., six teachers from the lower SES middle schools and four teachers from the higher SES middle schools participated in the stage of interviewing. In China, five teachers from the lower SES middle schools and four teachers from the higher SES middle schools participated in my interviews. As shown in Table 3.2, each interviewee's name (pseudonymous name), gender, years of teaching, certificate and interview number are provided.

Table 2

Background Information of Interviewees of the Study

		Gender	Years of teaching	Certificate	Interviewee number
U.S. higher SES					
	Baker	Female	20	Middle school math	U.S.-H-#1
	Carter	Male	10	Secondary math	U.S.-H-#2
	Denison	Female	9	Middle school math	U.S.-H-#3
	Edson	Female	6	Secondary math	U.S.-H- #4
U.S. lower SES					
	Fenning	Female	9	K-8 with concentration on middle school math	U.S.-L- #1
	Gerold	Female	10	Secondary math	U.S.-L-#2
	Haley	Female	15	Middle school math	U.S.-L-#3
	Iverson	Female	16	Secondary math	U.S.-L-#4
	Jordon	Male	6	Middle school math	U.S.-L-#5
	Kean	Male	5	Secondary math	U.S.-L-#6
Chinese higher SES					
	Zhao	Female	14	Middle school math	Chinese-H-#1
	Qian	Male	14	Middle school math	Chinese-H-#2
	Sun	Female	6	Secondary math	Chinese-H-#3
	Li	Female	18	Secondary math	Chinese-H-#4
Chinese lower SES					

	Zhou	Female	16	Middle school math	Chinese-L-#1
	Wu	Male	19	Secondary math	Chinese-L-#2
	Zheng	Female	5	Secondary math	Chinese-L-#3
	Wang	Male	10	Secondary math	Chinese-L-#4
	Feng	Female	8	Secondary math	Chinese-L-#5

Each interview in this study took approximately one hour to one hour and a half to finish. At the beginning of each interview, I asked for the participant's permission to take notes and use a digital audio recorder. Taking notes by hand was believed to be less obtrusive and less intimidating to some persons (Glesne, 1999). In addition, it helped to organize the researcher's thoughts and to respond to participants' responses during the interview. However, it may also interrupt the flow of the interview, because my attention might be focused on the struggle to keep up with the participants' talk. Thus, I used a digital audio recorder as the major recording instrument in order to really focus on listening to participants' responses.

Overall, questions in each interview were designed to cover the research questions, that is, questions regarding teachers' knowledge of functions and instruction, teachers' knowledge of functions and students, and teachers' knowledge of curriculum around functions. I began the interviews with a small set of open-ended questions (Hays & Singh, 2012) or "introducing questions" (Kvale, 1996, p. 133) that asked when they started teaching and in which schools were they employed as a teacher. Most of the interview time was devoted to probing the responses teachers already provided in their questionnaires and lesson plans. That is, using probing questions (Kvale, 1996). For example, I asked participants to elaborate on math tasks they gave in their lesson plans or

to illustrate their reasons for choosing a particular mathematical task. I designed my own interview questions based on studies of teachers' pedagogical content knowledge (e.g., Ball et al., 2008; Carpenter et al., 1989; Ma, 1999), qualitative research interview guides (e.g., Bogdan & Biklen, 2007; Foddy, 1993; Glesne, 1999; Hays & Singh, 2012; Kvale, 1996), and my previous experience of conducting interviews. The interview protocols (both in English and in Chinese) are provided in Appendix C.

As Glesne (1999) pointed out, you have to always "leave the door open to return" (p. 68) after your data collection. The completion of interviews was not equal to the "saying-goodbye" to the field. After completing my interviews, I revisited four teachers, through internet (i.e. email and Skype), to confirm or clarify some details obtained from the previous interviews.

Time lines for acquisition.

The interviews with the U.S. teachers were conducted over the period from early March to the end of April in the year of 2013. The interviews with the Chinese teachers were conducted over the period from mid-May to early-July in the year of 2013.

Pilot study

In order to determine if my questionnaire and interview questions worked as intended and what revisions I may need to make, I pilot tested my questions with some teachers before I formally started my study. Pilot work helped in revising the actual wording of questions (Oppenheim, 1966). Researchers should pilot-test their questions with people that are similar to those they plan to interview (Maxwell, 2005) or with people who are drawn from the actual group of participants in the study (Glesne, 1999). For this study, I piloted my questions with a few middle school mathematics teachers

from the Phoenix metropolitan area and the Beijing metropolitan area, respectively. I asked my friends who taught middle school mathematics (both in Arizona and in Beijing) to write lesson plans on introducing the concept of function and also asked them to give me responses to my questionnaire. In the pilot study, I got the following information: (1) whether teachers' responses to the questions really helped to address my research questions; (2) how the teachers felt about the questions - were they comfortable with answering these questions- and whether they made sense to them; and (3) determined how long an interview would really take for each teacher. Revisions were made after the pilot study.

Data Management

Creating contact summary sheets.

Creating contact summary sheets for all participants can help researchers identify information about the participants, capture their own reflections about the data and outline initial salient themes in the data collection process (Hays & Singh, 2012; Miles & Huberman, 1994). For this study, using contact summary sheets was very helpful for data management and data analysis. A contact summary sheet that included the teacher's name and background information was created for each teacher when they agreed to participate in this study. When I obtained a teacher's lesson plan and questionnaire response, I wrote my reflections about the data as well as my possible interview questions for that particular teacher on his or her contact summary sheet. Interview data and reflections on the interview data were later added to the contact summary sheet. Hence, for each teacher, I had a complete "profile". The contact summary sheets not only served as a useful tool for data management, but also were helpful for later data analysis. In fact,

using contact summary sheets represented the initial stage of data analysis. The contact summary sheet template is provided in Appendix D.

Lesson plan and questionnaire response data

When participants sent their lesson plans and questionnaire responses to me via email, I saved these files within my personal computers. I, the researcher, was the only one who had access to and worked with the data. All identifying information in the lesson plans and questionnaire responses was removed before coding.

Interview data.

All the recorded interview data was transcribed verbatim as relevant. The interview data was coded together with the accumulated lesson plan and questionnaire response data. In accordance with the principles of the Institutional Review Board (IRB), I informed my participants that all identifying information would remain confidential. All interview records and written documents (i.e. lesson plans and questionnaire sheets) were securely stored in a locked drawer in the researcher's office. In addition, all identifying information of participants in the interview data was removed before coding. These transcripts were saved in my personal computers which were only accessible to me, the researcher. I also backed up these data files and stored them in a portable hard drive which was securely stored at my home.

Data Analysis

The data analysis of the present study was divided into two main parts. The initial data analysis emerged during the data collection process. During this phase, Emerson, Fretz, and Shaw's (1995) strategies of in-process analytic writing helped to develop some themes using commentaries on and asking questions about the transcriptions. The

commentaries and reflections on lesson plans, questionnaire responses, and interviews were helpful for recording ideas and thoughts for further data collection and data analysis. The second and main part of the data analysis was conducted after I collected all the data. In this phase, I personally transcribed all the interview data and also carefully reviewed all the transcripts for accuracy. I read all the transcripts, as well as lesson plans and questionnaire responses as a complete "data set". In this process, I used NVivo, a qualitative-data analysis software, to organize my texts and audio data files and to help code because of its fast, intuitive and comprehensive features.

Development of codes

The development of the codes of this study was a combination of deductive and inductive coding. Using deductive coding, I started my data analysis with some preliminary codes based on the conceptual framework and previous research findings. The upper level or the general categories of this study were drawn upon the three components of Shulman's (1986) construct of pedagogical content knowledge including: (1) teachers' instructional knowledge of functions; (2) teachers' knowledge of students' understanding of functions; and (3) teachers' knowledge of curriculum. Codes developed from these three domains were deductive codes. The lower level categories were those specific aspects within each component of PCK. For example, the construction of mathematical tasks within teachers' instructional knowledge was a middle level category. Drawing upon Stein and her colleagues' (e.g., Henningsen & Stein, 1997; Stein & Lane, 1996; Stein et al., 1996, 2000) work, mathematical tasks can be classified into four types. Codes which were developed from Stein's perspective of mathematical tasks were deductive codes. Inductive codes emerged from the lower level categories (and

subcategories) as the questionnaire, lesson plan and interview data were analyzed. In particular, inductive and line-by-line coding of the interview transcripts helped "to remain open to the data and to see nuances in it" (Charmaz, 2006, p. 50). This also echoed "open coding (Emerson et al., 1995; Strauss & Corbin, 1990)" which provided opportunities for all analytic possibilities to exist in the analysis process. For example, codes which were developed from participants' explanations of the construction of a particular mathematical task were inductive codes.

Within-case profiles and cross-case comparison

My goal to create a within-case profile for each teacher group was to portray teachers' current state of PCK for the topic of functions. The resultant display for each case of this study was conceptually ordered (Miles & Huberman, 2000), which allowed me to develop better understanding of each teacher group's PCK in terms of the components of Shulman's (1986, 1987) PCK. A cross-case comparison of teacher groups was conducted to compare the current state of middle school math teachers' PCK across countries and socio-economic statuses. I made a conceptually ordered display (Miles & Huberman, 2000) of the cross-case results as reflective of the three key components--instructional knowledge, understanding of student thinking, and curricular knowledge--of Shulman's (1986, 1987) PCK model. It helped me identify similarities and differences within each component of Shulman's PCK construct.

Trustworthiness

As Glesne (1999) pointed out, the credibility of research findings and interpretations depended on how a researcher established trustworthiness. Trustworthiness, or validity (Creswell, 2009) is a critical aspect of any research. For a

qualitative research, qualitative validity and qualitative reliability are the two major concerns that a researcher should take into account when she or he is validating the findings throughout the research process (Creswell, 2009).

Qualitative validity.

Qualitative validity is based on determining whether the findings are accurate from the view of the researcher, the participant, and the readers (Creswell & Miller, 2000). To ensure the validity of the findings, the present study adopted multiple strategies as follows.

Triangulation.

Triangulation indicates collecting data from diverse sources. In this study, multiple sources - lesson plans, questionnaires, and interviews – were triangulated to reduce the validity threats that may exist in any one particular method. For example, the use of self-administered questionnaires helped to reduce the influence of the researcher's bias on participants' response which is common in interviews.

Member checking.

Member checking (Lincoln & Guba, 1985), or "respondent validation" (Bryman, 1988, cited in Maxwell, 2006, p. 111), is when researchers consistently solicit feedback about the data and conclusions from their participants (Maxwell, 2006). Throughout my study, I sent parts of the data and data analysis to the participant teachers in both China and the U.S. to see whether they felt that these descriptions and analysis were accurate. In this way, I provided an opportunity for my participants to comment on the findings of my research.

Peer debriefing.

Lincoln and Guba (1985) pointed out that peer debriefing can help to enhance the accuracy of the account. I shared my findings with some of my colleagues – both inside and outside my own program - to elicit their opinions concerning my research. Feedback from peers helped me better review my data and analysis. I also consulted my committee members, asking their ideas and views concerning my data and analysis.

Qualitative reliability.

Qualitative reliability concerns the consistency in a researcher's approach throughout the research (Creswell, 2009). This study took the following procedures to ensure qualitative reliability or consistency in the approach: (1) checking my transcripts to avoid any mistakes made during transcription; and (2) keeping consistent definitions of codes and terms throughout the process of research (Gibbs, 2007, cited in Creswell, 2009, p. 190).

As mentioned above, I used multiple strategies to ensure the qualitative validity and reliability in order to establish the trustworthiness of my study. Generalizability was not the goal or focus of the present study, since it included only one big city with several schools and teachers from each country. However, the findings of this study may point to some patterns of the urban Chinese and U.S. middle school math teachers' pedagogical content knowledge. In addition, the specific documentation of the procedures in this study may provide opportunities for replications in similar contexts.

Researcher's Role

The role of the researcher in this study was not limited to that of the interviewer and interpreter. In this qualitative study, the researcher acted as "the research instrument"

(Maykut & Morehouse, 1994), which allowed for producing meaning from the data and understanding the complexities of human lives and relationships (Pickard, 2007). However, this inevitably brought the researcher's personal issues, such as the biases, into the qualitative research process (Creswell, 2009). Thus, I have to point out my potential biases related to my interest and experiences as well as to my culture background in this section.

The idea of the present study originated overtime from my long-term interests in teacher quality and teacher education, as well as the relationships between teacher education, teacher quality, and student achievement. As a student from China, I still remember how my mathematics teachers taught in the past in China. When I moved to the U.S. to study at ASU, I was exposed to how teachers taught mathematics, particularly through course readings about teacher knowledge. This exposure to how teachers taught in both countries was further coupled by my experiences working on teacher quality evaluation projects back in China and on STEM projects (Science, Technology, Engineering, and Mathematics) for minority students (e.g., Blacks and Latinos) in the U.S. These intersecting experiences and exposure, led me to conduct a country to country comparison of teachers' pedagogical content knowledge of mathematics functions.

There is, I believe, always a balance point between the emic and etic stand a researcher brings into his/her research. My Chinese background sometimes made me identify more with the Chinese teachers, because I went through the mathematics education system in which they now teach. I was, from this perspective, an insider. Thus it was possible that my view as a researcher, an outsider, may be blurred. It is, therefore, necessary to explicitly state my potential biases in order to attain as much objectivity as

possible, though I have implemented steps to ensure that the interpretation reflects the data.

Table 3

Data Collection Matrix

Research Questions	Data Sources	Interview Questions
RQ1: What instructional decisions do Chinese and U.S. middle school mathematics teachers make when planning a lesson to introduce the concept of function?	Lesson plans submitted by teachers; Interviews	e.g., #4, #5, #6.
RQ2: How do Chinese and U.S. middle school mathematics teachers understand students' thinking of functions?	Questionnaire; Interviews;	e.g., #7, #8.
RQ3: What curricular knowledge on the topic of functions do Chinese and U.S. middle school mathematics teachers have?	Questionnaire; Interviews.	e.g., #9, #10; #11; #12.

CHAPTER 4

WITHIN-CASE RESULTS

In Chapter 4, I present within-case results for each teacher group (i.e. U.S. higher SES teacher group, U.S. lower SES teacher group, Chinese higher SES teacher group, and Chinese lower SES teacher group) of the study. The purpose of the cases is to provide a descriptive profile of middle school math teachers' pedagogical content knowledge (PCK) for the topic of functions. Each of the case profiles are organized based on the three key components – instructional knowledge, understanding of student thinking, and curricular knowledge - of Shulman's (1987) PCK model. The data sources for each of the case profiles include participants' responses to questionnaires, lesson plans on introducing the concept of function and semi-structured interviews.

Each case profile of teacher groups includes three sections reflective of Shulman's (1987) PCK conceptual framework. The first section presents teachers' instructional knowledge for the introduction of functions, with a focus on the goal of teaching, the construction of mathematical tasks and the use of representations. The second section presents teachers' knowledge of student understanding of functions. Two problems are included. One problem is about students' mistakes on function definition and the other is about students' mistakes on functional graph. The third section presents teachers' curricular knowledge on functions, including the use of instructional materials (textbooks in particular), lateral and vertical curriculum knowledge.

The U.S. Higher SES Middle School Math Teacher Group

Teachers' instructional knowledge for introducing the topic of functions.

In this section, I provide three sub-sections to describe the U.S. higher SES middle school math teachers' instructional knowledge for introducing the topic of functions. These subsections are the goal of teaching introductory class of functions, the construction of mathematical tasks and the use of representations of functions. In each subsection, a description of teachers' instructional decisions is presented based on the data collected from lesson plans, followed by teachers' explanations for their instructional decision, which are based on the data collected from semi-structured interviews. In total, ten introductory lesson plans were collected from the U.S. higher SES middle school math teacher group and four of the teachers in this group were interviewed.

The goals of teaching introductory class of functions.

In this study, U.S. higher SES middle school math teachers' goals of teaching introductory class of functions covered seven aspects. They expected their students to be able to understand that function is one input with exactly one output; the concept of input/output; domain and range; slope/rate of change; linear functions; graphical representation of functions; and the vertical line test for identifying functions. See Table 4.

Table 4

Instructional Goals for the U.S. Higher SES Teacher Group

Goal	Number of response	Example	Teachers' explanations

Function as one input with exactly one output	10	e.g., "(Students will be able to) understand that function is an input-output relationship that has exactly one output for each input." (Lesson plan U.S.-H-#8)	Follow the standards.
Input/output	2	e.g., "(I will) teach that input is the number or value that is entered into a function, the x value. Output is the number or value that is the solution to a function, the y value." (Lesson plan U.S.-H-#6)	
Domain and range	6	e.g., "My students will be able to identify the domain and range of a relation." (Lesson plan U.S.-H-#2)	Help understand the definition of function.
Slope/rate of change	1	e.g., "Students will be able to understand the concept of slope or rate of change in this lesson." (Lesson plan U.S.-H-#9)	
Linear functions	1	e.g., "Students will be able to understand what linear functions are and to tell their differences from nonlinear functions." (Lesson plan U.S.-H-#9)	
Graphical representations	4	e.g., "(Students are expected to be able to) identify the basic function shapes that match specific situations." (Lesson plan U.S.-H-#5)	Help understand what a function is as one important form of representation.

Vertical line test	6	e.g., "(Understand that) if your vertical line would intersect with your graph at only one point, the graph displays a function." (Lesson plan U.S.-H-#4)	An easier way to identify functions; one more perspective to define a function.

In their introductory lesson plans, all the ten teachers explicitly stated that their students should be able to understand that function is a rule that assigns to each input exactly one output after the introductory class. For example, one teacher wrote in her lesson plan, "(Students will be able to) understand that function is an input-output relationship that has exactly one output for each input." (Lesson plan U.S.-H-#8)

Another teacher wrote in his lesson plan, "This lesson will introduce students to the idea of functions. Students will be able to define functions. A function is a rule that *ALWAYS* applies to an input value to provide the same output each and every time." (Lesson plan U.S.-H-#4)

When asked why they set this as a goal for the introductory class of functions, all the four teachers interviewed mentioned the standards, especially the *Common Core Standards*. For example, Ms. Denison said,

> It is stated in the Common Core Standards…See? 8.F.1. Students are expected to understand that a function is a rule that assigns to each input exactly one output. So, one of my goals for them is to make sure that they understand this… (Interview U.S.-H-#1)

Mr. Carter said in his interview, "We adopted the Common Core last year. … According to it (the *Common Core*), students are expected to understand that a function is

a rule that assigns each input exactly one output in the introductory class…" (Interview U.S.-H-#3)

Six teachers mentioned in their lesson plans that understanding domain and range is an important goal for their students. For example, one teacher wrote, "My students will be able to identify the domain and range of a relation." (Lesson plan U.S.-H-#2) Another teacher wrote, "My students are expected to understand what domain and range are and that a function is a relation in which every domain has exactly only one range value" (Lesson plan U.S.-H-#3)

When asked why they think domain and range are important for their students to understand, Ms. Baker said, "Understanding domain and range helps them understand what a function is. You can use mapping. Here, [The charts] help them understand the rule of correspondence between a domain and a range." (Interview U.S.-H-#2)

Similarly, Ms. Denison said,

> …Whereas for xx public schools they don't do domain and range until ninth grade for Algebra I. For my kids, I expect them to know what a domain is and what a range is now. I want them to know there is one more way to understand and identify a function…" (Interview U.S.-H-#1)

Four teachers stated in their lesson plans that they expected their students to be able to identify and to understand function graphs. For example, one teacher wrote that students were expected to be able to "identify the basic function shapes that match specific situations" (Lesson plan U.S.-H-#5)

One teacher wrote that students would be able to "understand the graph of a function is the set of ordered pairs consisting of an input and the corresponding output as well as to draw linear graphs on a coordinate plan" (Lesson plan U.S.-H-#6)

When asked why they expected their students to know and to understand function graphs in the introductory class, Mr. Carter explained,

> For middle school it's very basic. Identify the function. Is it a function or isn't it a function? I expect them to be able to do that by looking at graphs, tables, and plugging numbers into equations. Graphs are something visual… Kids like drawing and looking at graphs is a good way to help student identify is that a function or not? They will know that functions could be different shapes… (Interview U.S.-H-#4)

Six teachers mentioned in their lesson plans that students will be able to use the vertical line test to identify functions (and function graphs). For instance, Mr. Carter pointed out in his lesson plan that he expected his students to use the vertical line test to understand that, "If your vertical line would intersect with your graph at only one point, the graph displays a function." (Lesson plan U.S.-H-#4)

He also provided in his lesson plan some examples for using the vertical line test to identify whether a graph indicates a function or not. See Figure 4.

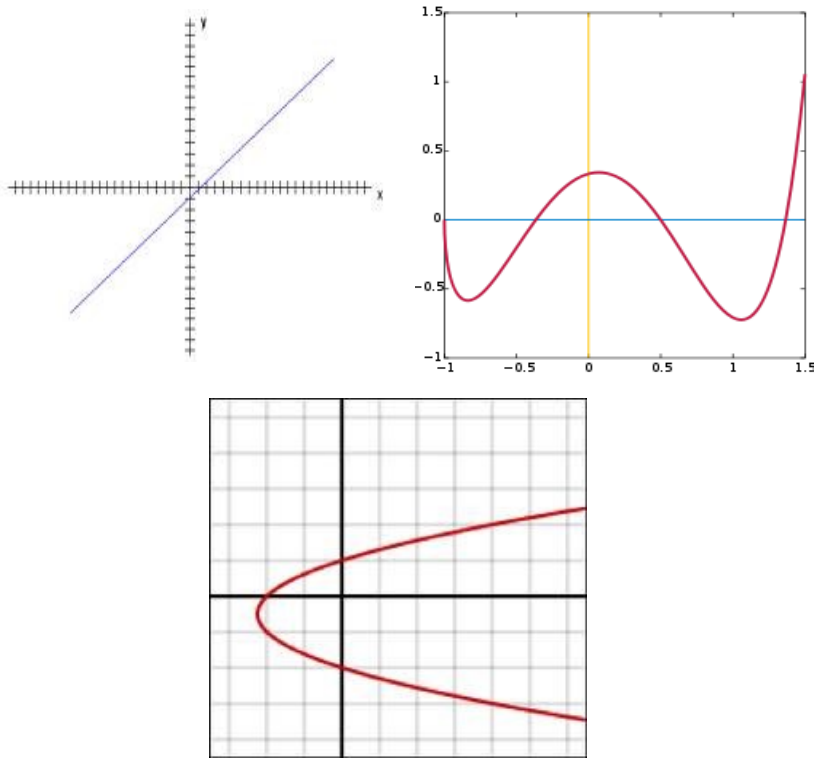


Figure 4 Using the vertical line test to identify functions by Mr. Carter.

When asked why he taught his kids the vertical line test in the introductory class, Mr. Carter explained in the interview,

> …If there's an easy way to do something, I teach the kids the easy way. I don't teach the hard way and then say, 'Okay, now that you know that, here's the easy way.' I teach the kids right off the bat the easy way. That way they can spend more time doing their homework and problems correctly. You teach something the difficult way and they struggle with the methods, how to get there, don't know why they're doing the steps. If I get them to the answer right away, all of a sudden they go, oh, I'm doing it right, and they wanna do more. That's just a function of the way I teach." (Interview U.S.-H-#3)

Ms. Edson, another teacher who pointed out the importance of teaching the vertical line test in the introductory class, also mentioned in the interview that using vertical line test "makes life easier". She said,

> …You then show them different types of graphs on the board and I would say, 'does this pass the vertical line test?' They would take out a pencil or whatever and they would just kinda, "No, it stops right there. It doesn't pass the vertical line test"…that skill was not difficult for them. That wasn't something that we had to spend an exorbitant amount of time. They pretty much got that right away as soon as we talked about it…it makes life easier. They'd have to be able to identify functions…" (Interview U.S.-H-#4)

Ms. Baker, who also included understanding the vertical line test in her instructional goals, explained in her interview that the vertical line test provided one more way to help students define a function, "When they do graphic on the coordinate plane and understand function has to pass a vertical line test, they can deal with linear and non-linear graphs and understand what a function means through graphs." (*Interview U.S.-H-#2*)

Two teachers wrote in their lesson plans that students were expected to understand input and output in the introductory class. For example, one teacher wrote, "[I will] teach that input is the number or value that is entered into a function, the *x* value. Output is the number or value that is the solution to a function, the y value." (Lesson plan U.S.-H-#6)

None of the teachers interviewed from this group included understanding input and output as an instructional goal in their lesson plans. Ms. Denison explained to me,

> …Understanding input and output are important to understand function…But, I mean, they already know what an input is and what an output is… We taught this a couple of weeks ago. They already know…they only need to use them in this class... (Interview U.S.-H-#1)

Only one teacher wrote in her lesson plan, "Students will be able to understand the concept of slope or rate of change in this lesson." (Lesson plan U.S.-H-#9) This same teacher also included, "Students will be able to understand what linear functions are and to tell their differences from nonlinear functions." (Lesson plan U.S.-H-#9)

The construction of mathematical tasks.

In total, twenty mathematical tasks were constructed in these teachers' introductory lesson plans. Based on Stein and her colleagues' prespective (Henningsen & Stein, 1997; Stein & Lane, 1996; Stein et al., 1996, 2000), mathematical instructional tasks can be classified into four categories based on their level of cognitive demand. Lower-level tasks include memorization and procedure without connection to understanding, meaning, or concepts, whereas higher-level tasks include procedures with connection to understanding, meaning, or concepts and doing-mathematics tasks. All the twenty mathematical tasks constructed in this teacher group are categorized as follows.

Table 5

Mathematical Tasks by Level of Cognitive Demand for the U.S. Higher Teacher Group

Cognitive demand of mathematical task	Example	Teachers' explanations
Lower-level demand		
Memorization (0)	N/A	

Procedures without connection to concepts/understanding/meaning (2)	e.g., Solve 1) x + 4 = 19; 2) y - 2.3 = 7.8; 3) 4z = 120; 4) w/9 = 8. (*Lesson plan U.S.-H-#9*)	
	Higher-level demand	
Procedures with connection to concepts/understanding/meaning (14)	e.g., The fastest-moving tectonic plates on Earth move apart at a rate of 15 centimeters per year. Scientists began studying two parts of these plates when they were 30 centimeters apart. How far apart will the two parts be after 4 years? (*Lesson plan U.S.-H-#7*)	Common core transition; make meaning of what you are learning; kids are smart.
Doing mathematics (4)	e.g., Draw a function card and write the rule on your sheet. Create and complete a function table with 3 input values. Graph your points on the coordinate plane. Cut out the three cards you just made and mix them together with your group members' cards. After you have switched tables, work as a group to sort out matching rules, tables and graphs. What do you find? Compare your matches with the group who made the cards to find out if you are correct. Be prepared to present to the whole class. (*Lesson plan U.S.-H-#4*)	Belief in kids' ability.

Note. Numbers in parentheses indicate number of responses.

Two of the twenty mathematical tasks were constructed at the level of cognitive demand that focused on procedures, without connection to concepts, understanding or meaning. One example of these mathematical tasks is:

Solve

1) X + 4 = 19;

2) Y - 2.3 = 7.8;

3) 4Z = 120;

4) w/9 = 8. (Lesson plan U.S.-H-#9)

Fourteen of the twenty mathematical tasks were at the level of cognitive demand that focused on procedures with connections to concepts or understanding or meaning. Some examples of these mathematical tasks are as follows:

The fastest-moving tectonic plates on Earth move apart at a rate of 15 centimeters per year. Scientists began studying two parts of these plates when they were 30 centimeters apart. How far apart will the two parts be after 4 years? (Lesson plan U.S.-H-#7)

Determine if the following relationship represents a function and explain why.

x	2	4	6	8
y	1	2	3	4

(Lesson plan U.S.-H-#8)

If the area of a triangle is 20 $cm^2$and the height is 10 cm, what is the base?

(Lesson plan U.S.-H-#10)

When asked the reason why they constructed these particular mathematical tasks, the teachers whose mathematical tasks focused on procedures with connections to concept, understanding, or meaning explained in their interviews as follows,

> We are transitioning to the Common Core… the old standards is multiple choice…Common Core is a lot like explain, explain your reasoning. Kids need to know why…they need to understand the meaning. We're doing a lot more of that in class… like how do you know your answer? Explain to me the reason, the meaning. (Interview U.S.-H-#2)
>
> The old standards were lower. The old standards just require kids to kind of learn and memorize and not really apply. Common Core makes them think… and apply. We've got to find more time for this…for them to think, to understand why, to connect to real meaning… (Interview U.S.-H-#1)
>
> Kids are smart. My kids are really smart. They can learn things. They can really understand. They should not be limited in a class that only requires them to memorize. (Interview U.S.-H-#3)
>
> Prior to Common Core, we used to just teach the concept. We never told them why you use it; when you're gonna use it, what it means. I think that's one of the things Common Core is going to address but we're not quite there yet. I want them know how you would use that, why you use it, why it's important. Make sense to me! (Interview U.S.-H-#4)
>
> My accelerated kids are used to understanding everything right away. They are good. They want to know why and how to use in real life… (Interview U.S.-H-#1)

Four of the twenty mathematical tasks constructed were categorized as "doing mathematics". For example, Ms. Edson constructed a "doing-mathematics" type of task in her lesson plan as follows,

> Draw a function card and write the rule on your sheet. Create and complete a function table with 3 input values. Graph your points on the coordinate plane. Cut out the three cards you just made and mix them together with your group members' cards. After you have switched tables, work as a group to sort out matching rules, tables and graphs. What do you find? Compare your matches with the group who made the cards to find out if you are correct. Be prepared to present to the whole class. (Lesson plan U.S.-H-#4)

Ms. Edson explained to me in the interview, "I have belief in my kids. I know they can do math. They are learning to not only to understand, but also to analyze, and to apply in different contexts…They can do higher level math." (Interview U.S.-H-#4)

The use of representations of functions.

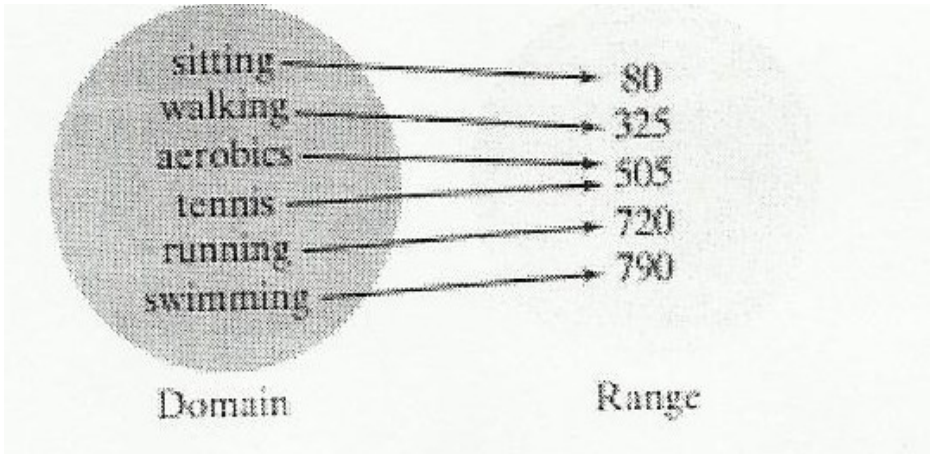
Teachers in this group used five forms of representations of functions in total in their lesson plans. These are graphs, algebraic functions, tables, ordered pairs and verbal. Examples for each form of representations are shown as follows,

Graphs:

For example,



(Lesson plan U.S.-H-#5)



(Lesson plan U.S.-H-#3)

Equations:

For example,

If f(x) =3x+1, find each of the following a) f(5) =_____ b) f(-4) =_____.(Lesson plan U.S. –H-#10)

Tables:

For example,

The following tables could NOT be functions because there is more than one *y*-value for each *x*-value.

X	Y
0	3
1	2
-1	2
2	-1
-2	-1

x	*f*(*x*)
-2	8
0	3
2	8
4	-2

(Lesson plan U.S.-H-#2)

Ordered pairs:

For example,

What is the domain and what is the range? Is this a function?

$$\{(1, 6)\ \ (-2, 5)\ \ (4, 17)\ \ (-3, 10)\}$$

(Lesson plan U.S.-H-#6)

Verbal:

For example,

Create a story to describe a function.

(Lesson plan U.S.-H-#8)

How do the teachers in this group use the five forms of representations of functions in their lesson plans? Specifically, all the ten teachers in this teacher group used graphs in their lesson plans (see Table 6). One teacher only used graphs in her lesson plan. Three of them used graphs and equations together in the introductory lesson plans. Three of them used graphs and tables together in the lesson plans. One used graphs, tables and equations together in the same lesson. One teacher used graphs and tables as well as

verbal representation together. One teacher used graphical representations together with ordered pairs in his lesson plan.

Table 6

The Use of Representations by the U.S. Higher SES Teacher Group

Form of representation	Number of response
Graph only	1
Equation and graph	3
Graph and table	3
Graph, table and equation	1
Graph, table and verbal	1
Graph and ordered pairs	1
Teachers' explanations: Graphs are important in understanding functions; kids like visual and graphs; and connecting to their prior knowledge.	

As Ms. Baker explained in the interview, "It is important to be able to use more than one representation. To be able to use multiple representations of functions is a goal in the *Common Core*. I want to show them different forms of representations…" (Interview U.S.-H-#1)

When asked about their attitudes towards using representations of functions, all the four teachers who were interviewed mentioned the importance of using graphs. For example, Ms. Baker said, "Graphs are important in helping kids understand what a function means…graphs are visual. Kids liked it because it was drawing." (Interview U.S.-H-#1)

Mr. Carter said in his interview,

> They (My students) have an activity giving them pictures and then they voted whether they thought it was a function or not… Kids like this activity because they like pictures and graphs. I can keep them engaged. It might be hard for them to understand what a function is if you use all the equations or algebraic functions in the first class…that is too abstract for them at the beginning even they will get there sooner or later. (Interview U.S.-H-#2)

Ms. Edson also explained,

> They spent a lot time working on identifying patterns from all kinds of pictures or graphs before getting to the concept of function…They feel comfortable if you give them graphs in learning the new topic 'cause they have been working on graphs for a long time…When we were doing functions, a lot of things we did were those that I show them on the coordinate plane, different kinds of shapes and things…a lot of different kinds of examples on the coordinate plane…Kids like drawing graphs. (Interview U.S.-H-#4)

Another point these teachers emphasized in their interviews is the conversion between graphic and other forms of representations of functions. For example, as Ms. Baker explained,

> I show them tables and graphs…I use equations too. I want them to know that functions can be represented in different forms and I want them to really see them in my class…I think it is important for them to be able to understand multiple representations of functions and to be able to translate a table of values to a graph or translate an equation to graphs, etc. It is difficult for kids actually… problems involving translations can be very hard for them… (Interview U.S.-H-#1)

Ms. Denison also said,

> [I used] the mapping, the table, and the graph. So they see it in three different formats. Not so much as an equation, but they also can see the equation too. Can you transfer between these different formats of functions? I ask them to do so. It is important… (Interview U.S.-H-#3)

Teachers' knowledge of student understanding of functions.

In this section, I provide detailed descriptions of teachers' response to two scenarios of students' mistakes. The first scenario is about students' mistakes in drawing function graphs through two given points on a coordinate plane. The second scenario is about students' mistakes in comparing two linear functional graphs on a coordinate plane. For each scenario, I describe teachers' knowledge of student understanding of functions from three aspects, based on their responses in the questionnaire. First, I describe the mathematical ideas that teachers think are important for students to correctly solve the math problem; second, I describe the "thinking" these teachers suggest their students might have leading to the mistake; and third, I describe teachers' approaches of correcting their students' mistake. In each of the aspects, a description of teachers' responses is presented based on the data collected from questionnaires, followed by some teachers' explanations collected from their semi-structured interviews. In total, eleven questionnaires were collected from the U.S. higher SES middle school math teacher group and four of the teachers in this group were interviewed.

Scenario 1.

A review of the scenario:

> If a student is asked to give an example of a graph of a function that passes

through the points A and B, she gives an example as shown in Figure 2. When asked if there is another answer for this question, she says "No".



Table 7

U.S. Higher SES Teachers Dealing with Students' Mistake on Drawing Graph(s)

Teachers' response (Number of response)	Example	Teachers' explanations
Mathematical ideas suggested to solve the problem		
Function definition: one input exactly with one output (7)	e.g., "They should understand what a function means. They should know that function is a rule assigning to each input exactly one output." (Questionnaire U.S.-H-#4)	
The vertical line test. (5)	e.g., "Students should be able to apply the vertical line test to correctly solve the problem." (Questionnaire U.S.-H-#5)	
Non-linear	e.g., "Knowing non-linear functions and	

functions (and graphs). (7)	graphs (e.g., Questionnaire U.S.-H-#2 & #6)"	
Linear function features. (5)	e.g., "Know linear function features." (e.g., Questionnaire U.S.-H-#2 & #7)"	
Student thinking or student misconceptions		
Connecting two points create a line. (5)	e.g., "They are thinking 'connecting two points create a line'." (e.g., Questionnaire U.S.-H-#2, #3, & #6)	Have no experience with non-linear functions; do not really understand what a function is. (See Interview U.S.-H-#1 & #2)
A function is a line (or a linear function). (6)	e.g., "They might be thinking 'a function is a line'." (e.g., Questionnaire U.S.-H-#5, #7)	
Approaches to correct students' mistakes		
Let students draw a couple of graphs and discuss which are functions using definition of function. (3)	e.g., "I think I would let students draw a couple of graphs by themselves. Then we discuss which are functions using the definition of function." (Questionnaire U.S.-H-#6)	It is necessary to really understand the concept of function. (See Interview U.S.-H-#2)
Let students draw a couple of graphs and discuss which are functions using the vertical line	e.g., "I would let students draw a couple of graphs by themselves and discuss with them which are functions using the vertical line test." (Questionnaire U.S.-H-#5)	The importance of the vertical line test in identifying functions. (Interview U.S.-

test. (2)		H-#3)
Expose students to a variety of functions in different representations they have seen. (2)	e.g., "I would provide them a variety of functions in different representations – graphs and tables and equations- they have seen. They would see lines like y =2x. They would see parabolas like $y=2x^2$. They would see curves like $y=x^3$, etc. They need to understand that functions come in different forms and different shapes." (Questionnaire U.S.-H-#1)	Students' lack of experience of working with nonlinear functions. (See Interview U.S.-H-#1)
Directly use counter-example of functions. (3)	e.g., "I would draw a parabola through the two given points. And then we would discuss if it is a function." (Questionnaire U.S.-H-#8)	
Make up stories for linear and nonlinear functions and graph them. (1)	e.g., "I would give my students some stories which include linear and nonlinear functions. I would ask them to distinguish between linear and non-linear functions from my stories. I then would ask them to try to graph them. They would understand that there are infinite graphs connecting two given points." (Questionnaire U.S.-H-#11)	

There are four mathematical ideas that teachers in this group think are important for students to correctly solve the problem: function definition; non-linear functions and graphs; the vertical line test; and linear function features.

Seven out of the eleven teachers in this group mentioned in their questionnaire responses that understanding the definition of function is important for students to correctly answer the question in the scenario. For example, one teacher wrote in her questionnaire response, "Knowing function as one input with exactly one output is important to correctly answer this question." (Questionnaire U.S.-H-#1) For another example, one teacher responded in the questionnaire, "They should understand what a function means. They should know that function is a rule assigning to each input exactly one output." (Questionnaire U.S.-H-#4) In addition, one male teacher wrote in his questionnaire, "This problem is actually checking whether students understand the definition of function and know how to identify functions." (Questionnaire U.S.-H-#2)

Seven of the teachers in this group pointed out the importance of "knowing non-linear functions and graphs (e.g., Questionnaire U.S.-H-#2 & #6)" and five teachers mentioned "knowing linear function features". (e.g., Questionnaire U.S.-H-#2 & #7)

The vertical line test, as mentioned by five teachers, was also regarded as one important mathematical idea that students need to know to answer the question correctly. For example, Ms. Edson wrote in her questionnaire response, "Students should be able to apply the vertical line test to correctly solve the problem." (Questionnaire U.S.-H-#5)

She explained to me in the interview,

> Once they know the vertical line test, they can use it to identify function graphs on a coordinate plane no matter [if] it is linear or non-linear…no matter whether they have seen the graphs before…or not…I mean they can simply draw a vertical line to cross the graph… (Interview U.S.-H-#4)

When asked what they think their students are thinking that leads to this mistake, two types of answers were found in these teachers' questionnaire responses. One is that, "They are thinking 'connecting two points create a line'."(e.g., Questionnaire U.S.-H-#2, #3, & #6) Six of the eleven teachers in this group wrote down this answer in their questionnaire responses. The other five teachers responded in their questionnaire that "They might be thinking 'a function is a line'."(e.g., Questionnaire U.S.-H-#5 & #7)

When asked why they think their students might have this thinking in the follow-up interviews, the four teachers interviewed gave me the following explanations. Ms. Baker and Ms. Denison pointed to students' lack of experience of working with non-linear functions. For example, Ms. Baker said in her interview,

> If this were my kid, she would make this mistake merely because we did not talk much about non-linear functions at middle school. They worked a lot on linear functions and graphs. They know connecting two points create a line and they equate functions with lines…we talked a lot on linear…not quadratic or cubic at this point. (Interview U.S.-H-#1)

Mr. Cater and Ms. Edson pointed to students' lack of real understanding of what a function is. For example, Mr. Cater said in his interview,

> …Yeah, connecting two points create a line, but that [is] only for linear functions. If they make this mistake, I think it might be that they actually do not grasp the concept of function…what a function really means…function is one input with exactly one output. It can be anything as long as it satisfies this condition. (Interview U.S.-H-#2)

When asked how they would help student correct this mistake in their questionnaire, teachers in this group came up with several different approaches.

Two teachers mentioned that they would choose to expose students to a variety of functions in different representations. For example, Ms. Baker wrote in her questionnaire response that,

> I would provide them a variety of functions in different representations – graphs and tables and equations- they have seen. They would see lines like y =2x. They would see parabolas like y=2x2. They would see curves like y=x3, etc. They need to understand that functions come in different forms and different shapes. (Questionnaire U.S.-H-#1)

She explained in her interview that,

> I would use this approach…they just lack the experience of working with non-linear functions. I will provide the opportunity for them to see different types of functions and graphs. Once they see…I mean they are exposed to different kinds of functions, they will know you can draw anything, lines, curves… between two points on a coordinate plane. (Interview U.S.-H-#1)

Three teachers wrote in their questionnaires that they would let students draw graphs and discuss with them which are functions using the definition of function. For example, Ms. Edson wrote in her questionnaire response, "I think I would let students draw a couple of graphs by themselves. Then we discuss which are functions using the definition of function. (Questionnaire U.S.-H-#6)

She explained in her interview why she chose to use this approach,

> I think understanding the definition of function is important for students to correctly solve the problem… I believe they probably lack a real understanding of function that leads to making the mistake. You really need to emphasize the rule that one input with exactly one output. (Interview U.S.-H-#4)

Mr. Carter, who pointed out the importance of function definition and students' lack of real understanding of the concept of function, wrote in his questionnaire response that, "I would choose to teach and discuss the definition of function in my class. Then we come back to solve this problem." (Questionnaire U.S.-H-#3)

He explained in his interview,

> I think it is necessary to teach the concept again. I think they may not really understand what a function is especially after we spend a big amount of time on learning linear functions. It is a time to pick up the concept again… (Interview U.S.-H-#2)

Two teachers wrote in their questionnaire responses that they would use the vertical line test in correcting this mistake. For example, Ms. Denison, who pointed out the importance of the vertical line test and students' lack of experience of working with non-linear functions, wrote in her questionnaire, "I would let students draw a couple of graphs by themselves and discuss with them which are functions using the vertical line test." (Questionnaire U.S.-H-#5)

Ms. Denison explained in her interview,

> I think sometimes it is important to ask kids to find out why they make mistakes…I do not need to point out saying, 'you are wrong, you have to do this this and that…' you know what I mean? They can do it by themselves. I taught

them the vertical line test at the first few classes, they forgot as we moved along. They just need to pick it up again. They are smart. Use the vertical line test. That's an easy way. (Interview U.S.-H-#3)

Two more types of correcting approaches were found in teachers' questionnaire responses. Three teachers wrote in their questionnaire responses that they would directly use counter-example of functions. For example, one teacher wrote, "I would draw a parabola through the two given points. And then we would discuss if it is a function." (Questionnaire U.S.-H-#8)

The last approach is, as one teacher wrote in her questionnaire response, I would give my students some stories which include linear and nonlinear functions. I would ask them to distinguish between linear and non-linear functions from my stories. I then would ask them to try to graph them. They would understand that there are infinite graphs connecting two given points. (Questionnaire U.S.-H-#11)

Teachers' approaches of correcting students' mistake in this group were then categorized into five types. Type 1: Three teachers chose to let students draw a couple of graphs by themselves and discuss which ones are functions using the definition of function. Type 2: Two teachers chose to let students draw a couple of graphs by themselves and discuss with the class which are functions using the vertical line test. Type 3: Two teachers would provide students a variety of functions in different representations they have seen. Type 4: Three teachers would directly use counter-example of functions. Type 5: One teacher would make up stories for linear and nonlinear functions and graph them.

Scenario 2.

A student is given the position vs. time graph as presented below. When asked to compare the speeds of the objects at time $t = 2$ sec., the student responds by saying that Object B is moving faster.

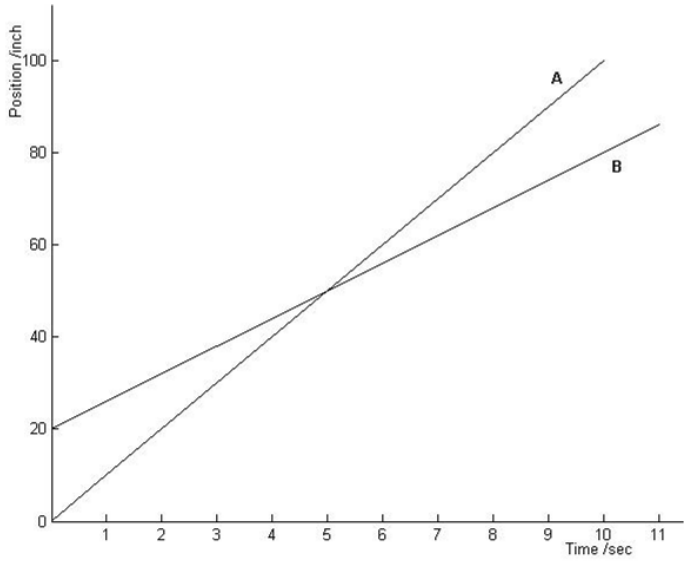


Table 8

U.S. Higher SES Teachers Dealing with Students' Mistake on Comparing Linear Functions on A Coordinate Plane

Teachers' response (Number of response)	Example	Teachers' explanations
	Mathematical ideas	
Understand the meaning of function graphs in real life. (8)	e.g., "They should know what that graph means and what the lines mean in real life. It is very common to see all kinds of 'moving', 'motion' in daily life. The lines can be anything	

	like a person walking, a car moving, an object slide, etc." (Questionnaire U.S.-H-#3)	
Slope/rate of change in graph and meaning. (7)	e.g., "It is actually checking students' ability of comparing rates of change or slopes when a problem asks to find out which one is moving faster." (Questionnaire U.S.-H-#6)	
Y-intercept (4)	e.g., "Students need to understand meaning of y-intercept. Starting points are key to correctly solving this problem. For a linear function graph, the y-intercept or the y-value when x-value equals zero determines the location of the starting point." (Questionnaire U.S.-H-#2)	
Comparison of numbers and graphs. (1)	e.g., "students need to know what comparisons between numbers mean and what comparisons between graphs mean" (Questionnaire U.S.-H-#7)	
Definition of function (1)	e.g., "They should understand the definition of function." (Questionnaire U.S.-H-#11)	
Formula of average speed. (1)	e.g., "[They need to know] the formula of average speed." (Questionnaire U.S.-H-#8)	
	Student thinking or student misconceptions	
B is higher at t=2	e.g., "[They might be thinking] B is	Careless

on the graph – B moves faster (6)	higher at t=2 on the graph so B moves faster." (e.g., Questionnaire U.S.-H-#1, #3, #, & #8)	mistakes; comparison of lines (and points) on a graph is a higher-level of math skill. (See Interview U.S.-H-#1 & #2)
B is higher on the graph (2)	e.g., "[They might be thinking] B is higher on the graph." (e.g., Questionnaire U.S. – H- #2, #5, #6)	
B is higher at the starting point. (2)	e.g., "[They might be thinking] B is higher at the starting point." (e.g., Questionnaire U.S. –H-#7, #9)	
The points on the graph is the speed (1)	e.g., "The students might be thinking 'speed is the points on the graph'" (Questionnaire U.S. –H-#11)	
	Correcting approaches	
Provide real-life examples to understand the relationship between distance, time and speed (1)	e.g., "I would provide real-life examples for them understand the relationship between distance, time and speed (Questionnaire U.S.-H-#8)	
Initiate a discussion on steepness and height of lines and how to determine them (2)	e.g., "Initiate a discussion on steepness and height of lines." (Questionnaire U.S. –H-#1, #6)	Help students learn systematically how to read this type of graph and how to make comparisons. (See Interview

		U.S.-H-#1)
Ask students to discuss the concept of slope/rate of change (3)	e.g., "[I would] ask them to tell me what slope/rate of change is. We will then discuss the concept of slope/rate of change." (Questionnaire U.S.-H-#9)	
Carefully read the graph again to understand the information given in the graph. (4)	e.g., "Carefully read the graph again to understand the information given in the graph." (Questionnaire U.S.-H-#2 & #3)	Kids are smart but they tend to make careless errors. (See Interview U.S.-H-#2)
Use a table of values to help compare. (1)	e.g., "[I would] ask them to make a table of values to help compare graphs." (Questionnaire U.S.-H-#10)	

There were a total of six mathematical ideas or concepts that teachers in this group thought were important for students to know in order to correctly solve this problem: the meaning of function graphs in real life; slope or rate of change in graph and meaning; y-intercept; comparison of numbers and graphs; definition of function; and formula of average speed as distance divided by time. Eight out of the eleven teachers in this group mentioned that understanding the meaning of function graphs in real life is important for students to correctly answer the question in the scenario. For example, Ms. Denison wrote in her questionnaire response,

> They should know what that graph means and what the lines mean in real life. It is very common to see all kinds of 'moving', 'motion' in daily life. The lines can be anything like a person walking, a car moving, an object slide, etc. (Questionnaire U.S.-H-#3)

Seven of the teachers in this group pointed out the importance of understanding slope or rate of change in graphs. As Ms. Edson wrote in her questionnaire, "It is actually checking students' ability of comparing rates of change or slopes when a problem asks to find out which one is moving faster." (Questionnaire U.S.-H-#6)

Four of the teachers in this group mentioned the student needed to understand what "y-intercept" means in order to correctly answer the question. Mr. Carter wrote his response,

> Students need to understand meaning of y-intercept. Starting points are key to correctly solving this problem. For a linear function graph, the y-intercept or the y-value when x-value equals zero determines the location of the starting point. (Questionnaire U.S.-H-#2)

One teacher wrote in her questionnaire response that, "Students need to know what comparisons between numbers mean and what comparisons between graphs mean" (Questionnaire U.S.-H-#7), and one teacher wrote that, "They should understand the definition of function." (Questionnaire U.S.-H-#11) Additionally, one teacher wrote, "[They need to know] the formula of average speed." (Questionnaire U.S.-H-#8)

When asked what they think their students might be thinking when solving this problem, four types of answers were found in these teachers' questionnaire responses. Six of the eleven teachers in this group wrote in their questionnaire responses, "[They might be thinking] B is higher at t=2 on the graph so B moves faster." (e.g., Questionnaire U.S. – H-#1, #3, #4, & #8) Two of the teachers in this group wrote in their questionnaire responses, "[They might be thinking] B is higher on the graph." (e.g., Questionnaire U.S.-H- #2, #5, & #6) Two teachers wrote in their questionnaire response, "[They might be

thinking] B is higher at the starting point." (e.g., Questionnaire U.S.-H-#7 & #9) One teacher wrote in her questionnaire response, "The students might be thinking 'speed is the points on the graph'."(Questionnaire U.S.-H-#11)

When asked why their students might think this way when solving the problem, the four teachers interviewed in this group provided the following explanations. Mr. Carter and Ms. Edson pointed to students' carelessness in their interviews. For example, Mr. Carter said in his interview,

> I do not think they are not able to solve this problem. They are higher-level kids. It is not difficult for them. They've seen this before. They know the concept and they know how to do distance, speed and time. They just do not read the graph carefully. And I know it happens all the time. They are not patient to read the problem, read the graph… (Interview U.S.-H-#2)

Ms. Baker and Ms. Denson pointed out in their interviews that student had this thinking because they lacked experience in doing comparisons between lines, graphs, and functions. For instance, Ms. Baker explained in her interview,

> Comparison is a higher-level of math skill for kids. First you need to figure out what is being compared and then how to compare. I think it is difficult for kids at this point. I think it is because they do not know how to compare that leads to their mistakes. When they say, "B is higher at t=2 so B moves faster", they are comparing the points of the lines or the heights of the lines. They are making the incorrect comparisons (Interview U.S.-H-#1)

When asked how they would help students correct his mistake, teachers in this group came up with several approaches in their questionnaire responses. Ms. Baker and

Ms. Edson wrote in their questionnaire response that they would "initiate a discussion on steepness and height of lines." (Questionnaire U.S. –H-#1 & #6) Ms. Baker said in her interview,

> I wanna use this approach mainly because I want them to learn systematically how to read this type of graph and how to make comparisons. The steepness of a line is determined by the slope or rate of change or the coefficient of the linear term. If we compare speed, we want to look at this. If this is a uniform motion, the rate of change should be constant. The height of a line is determined by the y-value of the line or the function. It can be applied to all kinds of motion. They need to learn it in order to avoid making this mistake. (Interview U.S.-H-#1)

Four teachers wrote in their questionnaire responses that they would ask students to carefully read the graph again. For example, Mr. Carter and Ms. Denison, who thought that their students made this mistake mainly because of carelessness, suggested asking students to "carefully read the graph again to understand the information given in the graph." (Questionnaire U.S.-H-#2 & #3) Mr. Carter explained to me in his interview,

> I know my kids. They know how to do it. They just tend to make careless errors. It happens. When they take my tests, a lot of kids make careless errors. They forgot to put the negative sign in front of something or they threw away a multiplication sign and they read the number as 43 instead of 4 times 3…For this problem. I will have to ask them to read the graph again and again to make sure they pay attention to all information… I have to tell them stories to let them know what careless errors can do for you. These errors can be disasters when they are in industries… (Interview U.S.-H-#2)

Additionally, three teachers stated in their questionnaire responses that they would ask students to discuss the concept of slope or rate of change. For example, one teacher wrote, "[I would] ask them to tell me what slope/rate of change is. We will then discuss the concept of slope/rate of change." (Questionnaire U.S.-H-#9)

Additionally, one teacher wrote in her questionnaire response that "I would provide real-life examples for them understand the relationship between distance, time and speed" (Questionnaire U.S.-H-#8) and one teacher wrote in her questionnaire response that "[I would] ask them to make a table of values to help compare graphs." (Questionnaire U.S.-H-#10)

Teachers' approaches of correcting students' mistake in this group were then categorized into five types. Type 1: Provide real-life examples for students to understand the relationship between distance, time and speed. Type 2: Initiate a discussion on steepness and height of lines and on how to determine them. Type 3: Ask students to discuss the concept of slope/rate of change for graphs. Type 4: Ask students to carefully read the graph again to understand the information given in the graph. Type 5: Use a table of values to help compare the graphs.

Teachers' curricular knowledge.

In this section, I provide a description of the U.S. higher SES middle school math teachers' curricular knowledge. Specifically, I first present these teachers' response to what instructional materials, textbooks, in particular, they use for teaching the topic of functions. The data source for this is the questionnaires collected from these teachers. I then present these teachers' explanations of how they use these instructional materials, textbooks, in particular, in the classrooms. The data source for this is the interviews.

Lastly, I present these teachers' lateral and vertical curriculum knowledge. Lateral curriculum knowledge here refers to teachers' interdisciplinary work between functions (or math) and other disciplines. Vertical curriculum knowledge here refers to teachers' familiarity with the past and present understanding of in school. The data source for both the lateral and vertical curriculum knowledge is the interviews.

Table 9

U.S. Higher SES Teacher Group's Curricular Knowledge

Instructional materials	Standards (9)
	Glencoe Algebra (3), *Holt Mathematics* (3), *McDougal Littell Algebra* (2), and *NCTM Illuminations* (1), etc.
	Teacher created materials (4); outside materials (7).
The use of textbooks	Use a few examples/problems
Functions and other disciplines	Math and science are intertwined. Try incorporation in class.
	Interdisciplinary work goes less intense as it used to be.
Functions and other topics in math.	Functions as part of the relationship system

Note. Numbers in parentheses indicate the number of responses.

Based on the questionnaire responses, nine out of the eleven teachers in this group wrote that they used standards such as the Common Core Standards as one instructional material (see e.g., Questionnaire U.S.-H-#1, #2, & #5). Frequently mentioned textbooks for teaching the topic of functions are *Glencoe Algebra, Holt Mathematics*, *McDougal Littell Algebra*, *and NCTM Illuminations*. Some teachers used one single textbook while some teachers used multiple textbooks in class. Four teachers mentioned that they did not have a textbook in class (see e.g., Questionnaire U.S.-H-#4 & #5). Instead, they used

teacher-created materials (see e.g., Questionnaire U.S.-H-#4). Additionally, more than half of the teachers in this group mentioned that they used outside materials such as internet resources as supplements to textbooks in class (see e.g., Questionnaire U.S.-#2, & #3).

According to teachers' interviews, when asked about their use of the textbooks, teachers explained that they used only some examples or problems from the textbooks rather than follow exactly the textbooks in class. For example, Ms. Baker said in her interview,

> …We use this which is required by our school and the school district. Every kid has one here in the classroom and one at home…I do use some of the problems in the textbooks, but I find some of those problem are not that good for my kid. I mean, for example, the numbers are too big and kids get frustrated and the computations that they think they can't do it. Or they don't have enough practice problems of a certain kinds that I see they struggle with…I write a lot of my own and type up a lot of my own work. These are my notes so I can do more examples. I create from my own resources… (Interview U.S.-H-#1)

She also adds, "…our AIMS scores are very high…our school is expected to maintain that level. A lot of pressure is on us to do that… I create problems just like those on the test. They will get more practice on that." (Interview U.S.-H-#1)

Similarly, Mr. Carter said in his interview,

> We are going to use the Common Core this year rather than Arizona State Standards…I did ask them to solve some problems from the Holt book, but I use a lot from my own. This year, because the curriculum has changed a lot, some of

the examples are still useful but some are not…I create my own PowerPoint presentation for my kids and I'd click through them… (Interview U.S.-H-#2)

When asked about the interdisciplinary collaboration between functions (or math) and other disciplines, teachers' responses varied according to their interview data. For example, Mr. Carter explained in his interview,

> We have lots of kids now that are –love math, hate science. Love science, hate math. I don't know how you can do that because they're so kind of intertwined, but a lot of kids do have that now. There's a division there. We talk to each other as teachers. We get together in teams and we talk about the kids and, well, what do you do to motivate them? …We talk about what motivates the kids and then try to see if we can incorporate that into the lesson for the kids. (Interview U.S.-H-#2)

Similarly, Ms. Denison said in her interview, "…We try to do some corporation with science teachers…Yeah, I ask our engineering teacher here to help me if I am going to teach a topic … We have labs here…" (Interview U.S.-H-#3)

Differently, Ms. Baker said in her interview,

> There is some interdisciplinary work going on but not as intense as it used to be… For about two or three years we did that. Now with the Common Core, rather than meeting with other teachers from other disciplines, we're now meeting strictly with math teachers and we're sharing with each other. (Interview U.S.-H-#1)

Similarly, Ms. Edson said, "We now do some interdisciplinary cooperation but not quite often. If we want to incorporate something teachers will help me, but there's no formal process for doing that…Not that much honestly…" (Interview U.S.-H-#4)

When asked to describe how they connected functions and related knowledge in students' math learning in school, teachers suggested they expected their students to understand functions as part of the relationship system according to their interviews. For example, Mr. Carter said,

> They [Students] learned to identify patterns, then different kinds of relations. Functions are a type of relations. You can make connections between all those topics…And some other related topics. For example, we learn linear relations before linear functions. Quadratic equations. They have seen these relations and equations before seeing these from the perspective of functions. They are connected to each other. Or, I'm teaching proportional reasoning. It's related to functions. It's related to variables. In order to get that, they need to know linear functions…all these lessons are related to each other. These topics are actually connected in middle school math learning. (Interview U.S.-H-#2)

In another interview, Ms. Edson said,

> …For instance, velocity. We teach this idea to them. This is taught before we teach functions. This is a relationship thing. When we see this again in the functions area, they already know this… they see function is actually a type of relations… You can always make these connections… (Interview U.S.-H-#4)

The U.S. Lower SES Middle School Math Teacher Group

Teachers' instructional knowledge for introducing the topic of functions.

In this section, I provide three sub-sections to describe the U.S. lower SES middle school math teachers' instructional knowledge for introducing the topic of functions. These subsections are the goal of teaching introductory class of functions, the

construction of mathematical tasks and the use of representations of functions. In each subsection, a description of teachers' instructional decisions is presented based on the data collected from lesson plans, followed by teachers' own explanations for their instructional decisions which are based on the data collected from semi-structured interviews. In total, ten introductory lesson plans were collected from the U.S. lower SES middle school math teacher group and six of the teachers in this group were interviewed.

The goals of teaching introductory class of functions.

In this study, the U.S. lower SES middle school math teachers' goals of teaching introductory class of functions covered eight aspects. They expected their students to be able to understand the definition of function as one input having exactly one output; the concept of input and output; the concept of variable; slope or rate of change; independent and dependent variable; graphical representation of functions; coordinate plane; and the vertical line test for identifying functions. See Table 10.

Table 10

Instructional Goals for the U.S. Lower SES Teacher Group

Goal	Number of response	Example	Teachers' explanations
Function as one input with exactly one output.	7	e.g., "Understand that a function from one set (input) to another set (output) is a rule that assigns to each input exactly one output." (Lesson plan U.S.-L-#4)	Standards.
Input/output	4	e.g., "Students will be able to identify input and output in a problem." (Lesson plan U.S.-L-#4)	Foundation of learning functions and

			student ability
variable	2	e.g., "[Students will] understand variables are letters that represent numbers. There are various possibilities for the numbers they can represent." (Lesson plan U.S.-L-#10)	
Slope/rate of change	1	e.g., "Students are expected to understand what rate of change (slope) means on a graph." (Lesson plan U.S.-L-#7)	
Independent and dependent variable	1	e.g., "They will understand independent variables and dependent variables." (Lesson plan U.S.-L-#9)	
Graphical representation	4	e.g., "Students are expected to understand how to draw function graphs and to be able to identify function based on graphs." (Lesson plan U.S.-L-#10)	Help understand what a function is as one important form of representations; student interest.
Coordinate plane	1	e.g., "Understand the coordinate plane and what x-axis and y-axis represents." (Lesson plan U.S.-L-#4)	Foundation of learning functions.
The vertical line test	3	e.g., "To be able to identify functions (linear and non-linear functions) using the vertical line	Helps identify functions.

test." (Lesson plan U.S.-L-#3)

In their introductory lesson plans, the most frequently mentioned goal (by seven teachers) was that students should be able to understand the function definition as each input having exactly one output. For example, one teacher wrote in her lesson plan, "Understand that a function from one set (input) to another set (output) is a rule that assigns to each input exactly one output." (Lesson plan U.S.-L-#4)

For another example, one teacher included in her lesson plan, "Students will understand that a function is a rule that assigns to each input exactly one output." (Lesson plan U.S.-L-#5)

When asked why this was an important goal for this introductory lesson, three teachers who were interviewed pointed to the Common Core Standards. As Ms. Fenning explained,

> It is important to understand the definition of function. This is one goal stated in the Common Core…We set up our teaching goal for kids based on what we are asked to…I mean we need to follow the standards, follow the curriculum in the school district. (Interview U.S.-L-#1)

Similarly, Ms. Iverson also said in her interview,

> It is one goal for students to understand, or at least to 'know' that functions is a rule which assign each input exactly one output in the Common Core…I think it is important for them to know this…they will use this to identify functions afterwards….(Interview U.S.-L-#4)

Four teachers mentioned in their lesson plans that being aware of the importance of function graphs is a goal for their students to achieve at the introductory class. For

example, one teacher wrote in the lesson plan, "Students are expected to understand how to draw function graphs and to be able to identify function based on graphs." (Lesson plan U.S.-L-#10) Another teacher wrote, "[Students will understand that] the graph of function is the set of ordered pairs consisting of an input and the corresponding output." (Lesson plan U.S.-L-#8)

When asked why they set graphical representations as an important goal in the first class, Ms. Haley said,

> Kids like graphs. They like drawing… I choose to bring in the new concept using something they are interested in. It simply makes life easier... It is important for them to be aware that graph plays a significant role throughout the learning of functions. It helps them understand what a function is. Graphical representation is one important form of representation of functions. They are gonna play with it all the time. (Interview U.S.-L-#3)

A similar explanation was found in Mr. Kean's response,

> Understanding the graphical representation of functions is definitely one of my goals for my kids in the first few classes. I need them to understand what a function means using graphs. I need them to know why some graphs we have seen as functions and why some are not…I know they like it at the very beginning because it is drawing. When it comes to translate between graphs and equations or…uh, or comes to understand information in a graph to solve a problem, they find it difficult. It can be very difficult, seriously. They will find it hard. I want them to realize at first that we are going to do 'math' with graphs. They are not

simply playing with picture. They have meanings and kids to explore those meanings… (Interview U.S.-L-#6)

Being able to understand the vertical line test was another important goal that three teachers explicitly stated that they expected their students to achieve in the introductory lesson. Specifically, they expected their students to be able to apply the vertical line test in identifying functions (and function graphs). For example, Mr. Jordon said in his lesson plan that he expected his students, "To be able to identify functions (linear and non-linear functions) using the vertical line test." (Lesson plan U.S.-L-#3)

He explained this goal to me in the interview,

It is important for them to understand this rule before they move on…the kids discovered that, yes, the vertical line has all the same domains so therefore, because it has all the same domains, they all line up. When they have the same domains you've got two that are the same, or more that are the same, so it won't pass the vertical line test. Once they saw that, it was like, oh, that's easy now…Once they got that-they have that a-ha moment and that discovery moment and they find out that it's easy. (Interview U.S.-L-#5)

Additionally, four teachers pointed out that they expected their students to understand the concepts of input and output in the introductory class. For example, Ms. Iverson explicitly stated in her lesson plan that, "Students will be able to identify input and output in a problem." (Lesson plan U.S. –L-#4)

She explained this goal in the interview,

I feel it is important to include this as a goal in my introductory lesson…I feel I need to. I know we have talked about these concepts before, uh, but I am not sure

if they still remember. I mean they will need to do math between input and output as they move along…functions. It is always input and output. I know my kid. They are, uh, at a lower level. I need to drill. I need to make it important for them. (Interview U.S.-L-#4)

Ms. Iverson also mentioned in here lesson plan that students were expected to, "Understand the coordinate plane and what x-axis and y-axis represents." (Lesson plan U.S.-L-#4)

She explained this goal to me in the interview,

It is about the concept of function…it is also about the related knowledge they learn before functions. Teaching is hard, as I said, because we have to do all the make-up along the way. Just bridge the gap... For example, I have to teach them multiplication and division because a few kids in my class still do [not] know how to do 15 divided by 4… Here, I need to include a coordinate plane…yes, they learned before. But as an important concept in functions, I have to make sure they understand what a coordinate means again in this class…It happens all the time. I mean, not just for this topic. We try to catch up every day… (Interview U.S.-L-#4)

Two teachers wrote in their lesson plans that students were expected to understand variables. For example, one teacher wrote, "[Students will] understand variables are letters that represent numbers. There are various possibilities for the numbers they can represent." (Lesson plan U.S.-L-#10) Additionally, one teacher wrote in her lesson plan, "Students are expected to understand what rate of change (slope) means on a graph." (Lesson plan U.S.-L-#7) One teacher wrote in his lesson plan, "They will understand independent variables and dependent variables." (Lesson plan U.S.-L-#9)

The construction of mathematical tasks.

Table 11

Mathematical Tasks by Cognitive Demand for the U.S. lower SES Teacher Group

Cognitive demand of mathematical task	Example	Teachers' explanations
	Lower-level demand	
Memorization (0)	N/A	
Procedures without connection to concepts/understanding/meaning (5)	e.g., Plug in the equations to see if they work: (1) 2x + 4y = 12 a) (1, 2); b) (-6, 0) (2) -3x + y = -13 a) (-3, -4); b) (4, -1). (Lesson plan U.S.-L-#2)	Warm-up activities as transitioning into math thinking; make up for prior knowledge.
	Higher-level demand	
Procedures with connection to concepts/understanding/meaning (10)	e.g., Given the input (-2, -1, 0, 1, 2) and output (4, 1, 0, 1, 4), please graph these values on a coordinate plane and determine if it is a function. Justify your answer. (Lesson plan U.S.-L-#7)	Common Core adoption; make introductory class fun and relevant
Doing mathematics (2)	See below	Challenge my kids.

Note. Numbers in parentheses indicate the number of responses that fall into that level.

A total of seventeen mathematical tasks were constructed in the introductory lesson plans of this teacher group. Five out of seventeen mathematical tasks constructed were at the cognitive demand level which is focused at the procedures without connection to concept, understanding or meaning. Some examples of these mathematical tasks are:

Plug in the equations to see if they work: (1) 2x + 4y = 12 a) (1, 2); b) (-6, 0), and (2) -3x + y = -13 a) (-3, -4); b) (4, -1). (Lesson plan U.S.-L-#2)

Given a function, complete the table of values:

f (x) = 2x – 3

x	y
3	
0	
3	

(Lesson plan U.S.-L-#3)

When asked why constructed this type of mathematical task, Ms. Iverson responded, "It is more like a warm-up activity. Just some simple questions, solve the equations, or simply plug in some values." (Interview U.S.-L-#4)

She also added,

> I find that to be really helpful, because when the kids come in, they've come in from English or science or P.E. or whatever and so they aren't necessarily thinking algebra when they walk in the door…You have to kinda direct everybody to the same thing and that's kind of a good way. This age group is a little wild." (Interview U.S.-L-#4)

Mr. Jordon also mentioned using this type of mathematical task as a "warm-up" for an introductory class. He explained this in her interview,

> These (the questions) are actually base-level knowledge. The kids I get are low. Most of them do not have the foundation…lack of those prerequisite skills…you know that…a lot of them still do not know how to do integers, proportions which

they should've learned in elementary…it's frustrating. We have to make up for prior knowledge. Otherwise they are gonna [be] stuck here. (Interview U.S.-L-#5)

Similar responses were found in Ms. Fenning's interview,

When I'm trying to teach liners to a kid who has a fifth or a sixth grade math level, I don't have a good reason not to teach the foundation. These simple questions, a lot of times calculations, are very basic that they need to know." (Interview U.S.-L-#1)

Ten out of the seventeen mathematical tasks were at the cognitive demand level which is focused on the procedures with connections to concepts, understanding or meaning. Some examples of these mathematical tasks are as follows:

Given the input (-2, -1, 0, 1, 2) and output (4, 1, 0, 1, 4), please graph these values on a coordinate plane and determine if it is a function. Justify your answer. (Lesson plan U.S.-L-#7)

Watch three VIDEO clips (these are a flying airplane, being poured water, and a moving car). In the videos we watched, what do you think the inputs may have been? What was the outcome for each? What do you think the relationship was between each input and output? (Lesson plan U.S.-L-#4)

Do the jumping jack experiment in groups of four (one jumper, one timer, one counter and one recorder). For each jumper, prepare a table for recording the total number of jumping jacks for every 10 seconds, up to a total of 2 minutes (120 seconds). Use the table of your jumping jack data to answer these questions: How did the jumping jack rates (the number of jumping jacks per second) in your

group change as time passed? How is this shown in your tables? (Lesson plan U.S.-L-#5)

When interviewed why they constructed these types of mathematical tasks, Ms. Gerold said tome in her interview, "*The Common Core* will not test kids simple calculation…it asks more…it ask for the understanding of the meaning…" (Interview U.S.-L-#2)

Mr. Jordon explained,

> You have to make connections…these kids are not that interested in learning some abstract stuff. It is just hard for them. You have to make it fun, make it kinda relevant. You gotta keep these middle school kids moving. They can only stay on probably one task for 15 minutes. You want them to learn something in that limited time. You gotta make it fun…You have to make meaning. You have to make it interesting for these kids… They are lack of intrinsic motivation to some extent… (Interview U.S.-L #5)

Ms. Fenning explained in her interview,

> We do an activity to introduce it…I'm not a firm believer in paper/pencil math. I feel my kids learn by doing. I want them to do those hands-up activities, to stand up, to go outside…These are things that kids will remember doing and connecting to math that they won't remember in paper and pencil…They'll remember when they did linear equations that they had to go out or when they did slope they had to go out and measure stairs is rise over run.

Oh, they had fun…it opened up a lot of dialogue and conversation about what real-life function looked like. Kids think of mathematics as this abstract thing that has not meaning to their real life. (Interview U.S.-L-#1)

Two out of the seventeen mathematical tasks were constructed at the "doing-mathematics" cognitive demand level. See the following example of "doing-mathematics" task from this group:

Here are two examples of functions: $3x+2$ and x^2. Fill in the function table and label the columns as shown:

x	Rule	y	x,y

Graph the x,y values on a coordinate plane. What do you notice about the input and output values? When you graph it, what do you notice? Is this a function? Why or why not? Write down your own rule between x and y. Fill in the table and graph on a coordinate plane. What do you notice about the input and output values? Is this a function? Why or why not? What are the similarities/differences between your own rule and the functions given above? (Lesson plan U.S.-L-#6)

Ms. Haley, who constructed this mathematical task, explained in her interview,

I know I may get upset about how my kids work on it, but I still want to challenge them…I mean my kids, at least they try, they try on these problems. I am building the skills for them. As long as I can see they're moving along, I feel good as a teacher. (Interview U.S.-L-#3)

The use of representations of functions.

Teachers in this group used five forms of representations of functions in their introductory lesson plans. These five forms were tables, graphs, equations, verbal, and ordered pairs. Examples for each form of representations in this group are as follows,

Tables:

For example,

When the x value has exactly one output we can say that the relationship between the input and output is a function.

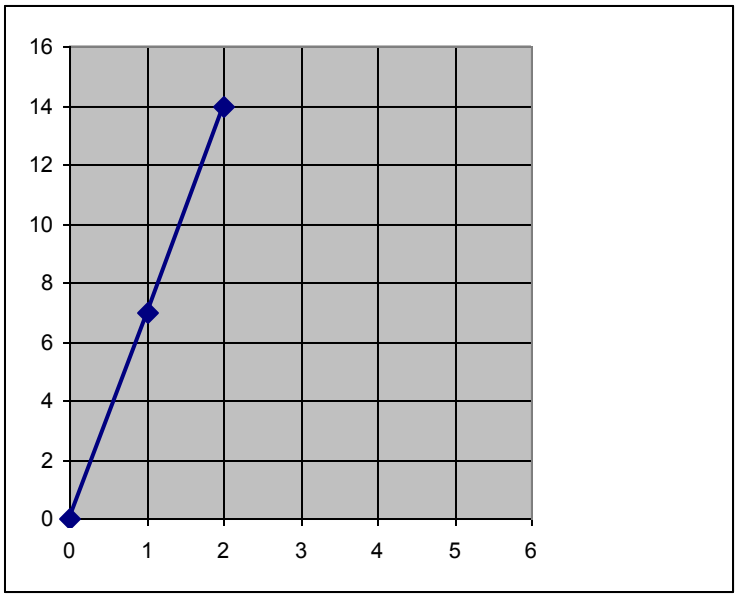
x	1	2	3	4	5	x	1	2	2	3
y	2	4	6	8	10	y	3	4	5	3

Relationship in the left columns is function because exactly one input per output, and relationships in the right columns are NOT functions because 2 has 2 different outputs. (Lesson plan U.S.-L-#4)

Graphs:

For example,

We can create our graph by plotting the three ordered pairs, (0, 0); (1, 7); (2, 14).



(Lesson plan U.S.-L-#3)

Equations:

For example,

Write the equation in function form

1. 4x + 2y =8, we get y= -2x +4;

2. 3x - 4y = 7, we get y=3/4x -7/4;

3. –x +1/2y= 3, we get y=-2x +6. (Lesson plan U.S.-L-#10)

Ordered pairs:

For example,

Determine if this is a function: (-2,4), (-1, 1), (0, 0), (1, 1), (2, 4). (Lesson plan U.S.-L-#8)

Verbal:

For example,

[After watching three video clips]

A FUNCTION is a rule that establishes a relationship between two quantities or things called INPUT and OUTPUT.

In the videos we watched what do you think the inputs may have been?

The flight of the airplane, the water being poured, and the car moving.

What was the outcome for each?

The height of the airplane decreased as time increased; the level of water increased as time increased; the distance of the car increased over time.

(Lesson plan U.S.-L-#2)

How did teachers in this group use these forms of representations of functions in their introductory lesson plans? Specifically, see Table 11, among the ten teachers whose lesson plans were collected in this group, all of them mentioned graphs and tables in their lesson plans. Four of the teachers used tables and graphs only in their lesson plans. One teacher used tables, graphs, and verbal representations in her lesson plan. One teacher used equation, graphs and tables together in her lesson plan. Three teachers used mappings, graphs and tables in the lesson plans. One teacher used tables, graphs, equations and verbal representations in his lesson plan. As Mr. Jordon explained, to be able to use multiple representations is highly encouraged. He said, "A student is encouraged to know tables, graphs, equations and verbal… [The] Common Core states that students are expected to use multiple representations of functions…." (Interview U.S.-L-#5)

Table 12

The Use of Representations by the U.S. Lower SES Teacher Group

Form of representation	Number of response

Table and graph.	4
Table, graph, and verbal.	1
Equation, table and graph.	1
Ordered pairs, graph, and table.	3
Table, graph, equation, and verbal.	1
Teachers' explanations:	
Graphs and tables are important in understanding functions; kids like visual, tangible things.	

In the interviews, when asked to explain their attitudes towards the use of representations of functions, teachers in this group provided two major explanations. Tabular and graphical representations, especially graphical representation, are important in the introductory lessons. As Ms. Fenning explained,

> We use tables and graphs…like this one…I show them a table of values and its graph. Then see consistent increases in X, consistent decreases in two. Then they should've sort of remembered that's gonna be linear. If this is always increasing by one, and this is always decreasing by two, I'm gonna get a line. They're not using the equation to get there. They are using this. (Interview U.S.-L-#1)

Ms. Gerold said in her interview,

> I mean, uh, they had to write three word problems dealing with linear functions. I didn't call them linear functions until after the fact, when they're graphing them. They're drawing them. They're using the tables. They're looking at the relationships from the x to the y's, the intercepts. What do the intercepts mean? Yeah. I try to bring in the vocab after they've constructed the information from the graph… (Interview U.S.-L-#2)

Mr. Jordon also had a similar idea. He explained,

> I use tables and graphs a lot in my class especially in the first few classes… I give them a word problem, and ask them to fill in the table of values using the information in the problem. I then ask them to use this table to draw a graph on a coordinate plane. It is very visual for my kids to understand what change means, what correspondence means in a relation…what the relationship will look like. They are more engaged when drawing pictures… (Interview U.S.-L-#5)

Teachers seemed to avoid or place less emphasis on algebraic functions because of its ability to complicate the lesson. For example, Ms. Iverson said,

> It is difficult for my kids to understand what a function is through the algebraic perspective…it is abstract. Not like graphs which seem more tangible and visual for kids. They do not necessarily need to use equations to understand the concept. I mean I know they will have to deal with equations sooner or later, but for now I do not expect them to do so… I do not want to make their life harder at the beginning. (Interview U.S.-L-#4)

For another example, Gerold said,

> I can't make an equation for it. No we do not ever talk about that. We're just saying, 'look at this. This is a representation. What's happening? What's the story? What's happening here?' We're not asking them to define an equation with it. Basically, what I'm saying is, kids are gonna have a misrepresentation that I should have an equation for this. But this is not the fact. (Interview U.S.-L-#2)

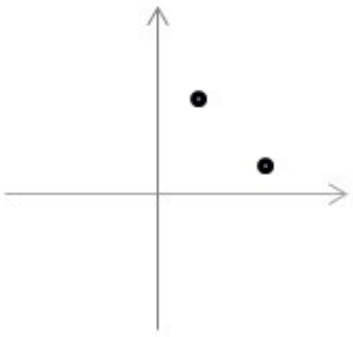
Teachers' knowledge of student understanding of functions.

In this section, I provide detailed description of teachers' response to two scenarios of students' mistakes. The first scenario is about students' mistakes on drawing function graphs through two given points on a coordinate plane. The second scenario is about students' mistakes in comparing two linear functional graphs on a coordinate plane. For each scenario, I describe teachers' knowledge of students' understanding of functions from three aspects, based on their responses in the questionnaire. First, I describe the mathematical ideas that teachers think are important for students to correctly solve the math problem; second, I describe the "thinking" these teachers suggest their students might have leading to the mistake; and third, I describe teachers' approaches to correcting their students' mistakes. In each of the aspects, a description of teachers' responses is presented based on the data collected from questionnaires, followed by teachers' explanations collected from their semi-structured interviews. In total, twelve questionnaires were collected from the U.S. lower SES middle school math teacher group and six of the teachers in this group were interviewed.

Scenario 1.

A review of the scenario:

If a student is asked to give an example of a graph of a function that passes through the points A and B, she gives an example as shown in Figure 2. When asked if there is another answer for this question, she says "No".



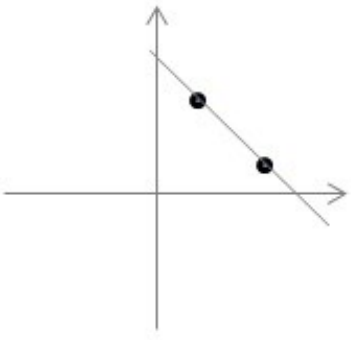


Table 13

U.S. Lower SES Teachers Dealing with Students' Mistake on Drawing Functional Graph(s)

Teachers' response	Example	Teachers' explanations
	Mathematical ideas	
Function definition: one input exactly with one output (7)	e.g., "Knowing function as one input with exactly one output is important for students to correctly answer the question." (Questionnaire U.S.-L-#3)	
The vertical line test (3)	e.g., "[Students need to know] the vertical line test to identify function and function graphs." (Questionnaire U.S.-L-#3)	
Non-linear graphs (4)	e.g., "Know nonlinear graphs." (e.g., Questionnaire U.S.-L-#2 & #5)	

Linear function features (2)	e.g., “Knowing linear function features.” (e.g., Questionnaire U.S.-L-#7)	
Input/output tables (3)	e.g., “It is important to be able to make input/output tables to help identify functions.” (e.g., Questionnaire U.S.-L-#7, #10)	
Slope or rise over run (3)	e.g., “Understand slope or rise over run (e.g., Questionnaire U.S.-L-#11)”.	
Equation determines the shape of graph (1)	e.g., “Know that an equation determines the shape of a graph.” (Questionnaire U.S.-L-#10)	
	Student thinking or student misconceptions	
Connecting two points create a line (6)	e.g., “[They might be thinking] connecting two points create a line.” (e.g., Questionnaire U.S.-L #1, #2, & #4)	No experience with non-linear functions; do not really understand what a function is (See Interview U.S.-L-#2, #3)
A function is a line (a linear function) (4)	e.g., “[The student might be thinking] function is a line.” (e.g., Questionnaire U.S.-L-#3, #5, & #11)	

Do not understand the problem (2)	e.g., "Students do not really understand the problem." (Questionnaire U.S.-L-#8)	
	Correcting approaches	
Let students draw a couple of graphs and discuss which are functions using the vertical line test (2)	e.g., "[I would] let students draw a couple of different graphs through the two points. We then discuss which ones are functions and which ones are not using the vertical line test." (Questionnaire U.S.-L-#5)	The vertical line test is really important for identifying functions. (See Interview U.S.-L-#4)
Expose students to a variety of functions in different representations they have seen (1)	e.g., "Expose students to different types of functions." (Questionnaire U.S.-L-#8)	
Directly use counter-example of functions (5)	e.g., "[I would] give a quadratic function, such as $y= 3x^2+2x-5$. Graph them on the coordinate plane. It can connect two points as a line does." (Questionnaire U.S.-L-#1)	Easy and straight for kids. (See Interview U.S.-L-#1)
make up stories (2)	e.g., "Come up with stories of functions and graph these functions." (e.g., Questionnaire U.S.-L-#10)	

Reteach the definition of function (2)	e.g., "I think I would teach them again the definition of function. Make sure they know what 'one input with exactly one output' means and how it is presented in a graph." (Questionnaire U.S.-L-#7)

Note. Numbers in parentheses indicate the number of responses.

In this group, seven mathematical ideas were mentioned in teachers' questionnaire responses for students to correctly solve the problem: function definition; non-linear functions and graphs; the vertical line test; linear function features; input and output tables; slope or rise over run; and equation determines the shape of graph.

Seven out of the twelve teachers in this group mentioned in their questionnaire response that the importance of definition of function. For example, one teacher wrote in her questionnaire response, "Knowing function as one input with exactly one output is important for students to correctly answer the question." (Questionnaire U.S.-L- #3)

A similar response was found that, "This is a type of question testing students' ability of identifying functions on graphs. They should know what a function is." (Questionnaire U.S.-L-#6)

Three teachers mentioned the importance of knowing the vertical line test. For example, one teacher wrote in his questionnaire response, "[Students need to know] the vertical line test to identify function and function graphs." (Questionnaire U.S.-L-#3)

Three teachers mentioned in their questionnaire responses that "it is important to be able to make input and output tables to help identify functions" (e.g., Questionnaire

U.S.-L-#7 & #10) and three teachers mentioned in their questionnaire responses that it was important to "understand slope or rise over run." (e.g., Questionnaire U.S.-L-#11)

Two of teachers in this group pointed out the importance of "knowing linear function features" (e.g., Questionnaire U.S.-L-#7) and four teachers mentioned in their questionnaire responses that it was important to "know nonlinear graphs." (e.g., Questionnaire U.S.-L-#2 & #5)

Additionally, one teacher mentioned in her questionnaire that, "Know that an equation determines the shape of a graph." (Questionnaire U.S.-L-#10)

When asked what would potentially lead students to make this mistake, teachers provided the following answers in their questionnaire responses. Six of the twelve teachers in this group wrote, "[They might be thinking] connecting two points create a line." (e.g., Questionnaire U.S.-L-#1, #2, & #4)

Five of the twelve teachers in this group wrote, "[The student might be thinking] function is a line." (e.g., Questionnaire U.S.-L-#3, #5, & #11)

Two teachers in this group wrote in their questionnaire responses that "students do not really understand the problem," (Questionnaire U.S.-L-#8) or "students do not understand the question asked in the problem." (Questionnaire U.S.-L-#10)

When asked why they think their students might have this thinking in the follow-up interviews, the six teachers interviewed provided the following explanations. Ms. Fenning, Ms. Gerold, Ms. Iverson and Mr. Jordon pointed to students' lack of experience of working with non-linear functions. For example, Ms. Gerold said in her interview,

> I think it is the misconception that 'connecting two points create a line' that leads to students' mistake…and how do they obtain this misconception? I think it might

be our fault. I mean we did not give them chance to see different functions. They played with linear, linear and linear… we introduced quadratic relations but did not spend much time on it. For eighth grade, linear is all they need to be familiar with…they forgot there were many non-linear functions and functions come in many shapes. (Interview U.S.-L-#2)

Ms. Haley and Ms. Kean pointed to students' lack of real understanding of what a function is as the most important reason. For example, Ms. Haley said in her interview,

I do not know if they really understand the concept of function. I doubt [it]. We did not spend much time on introducing the concept of function. I did ask them to identify some functions, but then we moved to linear functions. They probably totally forgot the definition of function. (Interview U.S.-L-#3)

When asked how they would help students correct this mistake, teachers in this group came up with some approaches in their questionnaire responses as follows. Five teachers in this group wrote that they would directly provide counter-examples of functions for their students. Ms. Fenning, Ms. Haley and Ms. Kean used this approach. For example, Ms. Fenning wrote in her questionnaire response, "[I would] give a quadratic function, such as $y= 3x^2+2x-5$. Graph them on the coordinate plane. It can connect two points as a line does." (Questionnaire U.S.-L-#1)

When asked why they would correct students' mistakes in this way in their interviews, they explained as follows. Ms. Fenning said,

I think it is the easiest way to correct their misconception. You give them some counter-examples. They know there are a lot [of] non-linear functions going

through the points given in the graph. [It is] very straight for my kids… (Interview U.S.-L-#1)

Ms. Haley explained in her interview, "As I said, I doubted if they really understand what a function is. I give them a lot of non-linear functions and graph as an opportunity to look at the concept of function… to see what a function is." (Interview U.S.-L-#3)

Ms. Kean, who thought her students may not understand what the problem asked, explained her answer in the interview,

> You know they may not know what the question is asking for…it is asking for drawing function graphs between two given points not asking for drawing lines. They do not see it. If I give them a lot of graphs, they will get it because they can see different curves between the two points. They will understand what the question really asks for… (Interview U.S.-L-#6)

Two teachers, Ms. Gerold and Ms. Iverson, chose to let students draw a couple of graphs and discuss which are functions using the vertical line test. For example, Ms. Iverson wrote in her questionnaire response, "[I would] let students draw a couple of different graphs through the two points. We then discuss which ones are functions and which ones are not using the vertical line test." (Questionnaire U.S.-L-#5)

Ms. Iverson explained in her interview,

> I would ask them to draw a couple of vertical lines between the two points…Vertical line test is really important for identifying functions and once they grasp it, they will not forget. They know as long as the graph passes the vertical line test, no matter it is linear or not, it is a function… They can draw a

random graph, an unusual curve, as long as it intersects the vertical line only once… (Interview U.S.-L-#4)

Two teachers mentioned in their questionnaire responses that they would reteach the definition of function in class. For example, Mr. Jordon, who mentioned the importance of understanding the definition, wrote in his questionnaire response, "I think I would teach them again the definition of function. Make sure they know what 'one input with exactly one output' means and how it is presented in a graph." (Questionnaire U.S.-L-#7)

Additionally, two teachers wrote in their questionnaire response that they would "come up with stories of functions and graph these functions," (e.g., Questionnaire U.S.-L-#10) and one teacher wrote that she would "expose students to different types of functions." (Questionnaire U.S.-L-#8)

The approaches of correcting students' mistake were then categorized into five types. Type 1: Let students draw a couple of graphs by themselves and discuss which ones are functions using the vertical line test. Type 2: Provide students a variety of functions in different representations. Type 3: Directly use counter-example of functions. Type 4: Make up stories for linear and nonlinear functions and graph them. Type 5: Use equations of non-linear functions and graph them on the coordinate plan.

Scenario 2.

A student is given the position vs. time graph as presented below. When asked to compare the speeds of the objects at time $t = 2$ sec., the student responds by saying that Object B is moving faster.

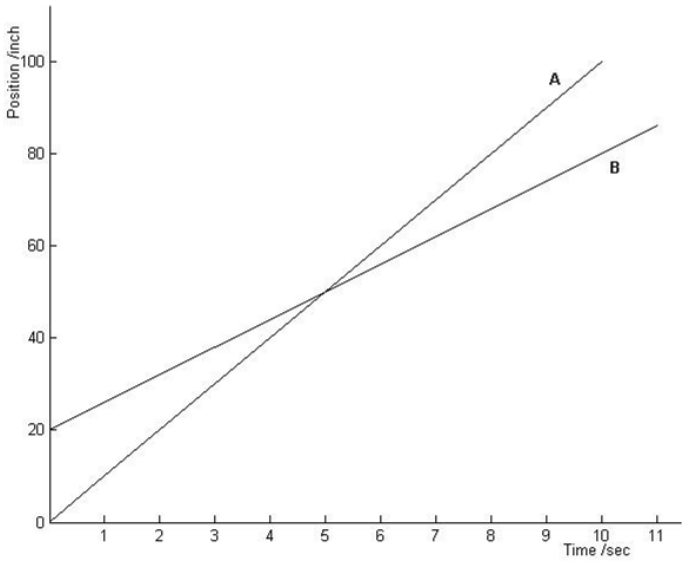


Table 14

U.S. Lower SES Teachers Dealing with Students' Mistake on Comparing Linear Functions on A Coordinate Plane

Teachers' response	Example	Teachers' explanations
	Mathematical ideas	
Meaning of x-value on the graph.(2)	e.g., "[Students need to know] the meaning of x-values on the graph." (e.g., Questionnaire U.S.-L-#11)	
Slope/rate of change in graph and meaning.(9)	e.g., "In a uniform motion, the speed is the slope of each line. Comparing speed is comparing slope actually." (Questionnaire U.S.-L-#1)	
y-intercept.(4)	e.g., "Pay attention to the y-intercept and the relationship between the starting point and y-intercept. They are actually the same thing even they	

	are in different forms." (Questionnaire U.S.-L-#3)	
Comparison of numbers and graphs.(2)	e.g., "[Students need to understand] the comparisons of numbers of graphs." (e.g., Questionnaire U.S.-L-#5)	
Meaning of distance, speed and time and relationship between them.(2)	e.g., "[They need to know] the meaning of distance, speed and time and their relationship. Plus, they need to find the information on the graph. For example, do they know the x-axis indicates the time? Do they know the y-axis indicates the distance the object moves?" (Questionnaire U.S.-L-#6)	
	Student thinking or student misconceptions	
B is higher at t=2 on the graph – B moves faster.(2)	e.g., "[Students might be thinking] B is higher at t=2 on the graph which means B moves faster." (Questionnaire U.S.-L- #3 & #8)	Lack of skills in comparison on a graph; do not know what the graph really means. (See Interview U.S.-L-#3,#4)
B is higher on the graph.(6)	e.g., "[Students might be thinking] B is higher on the graph." (e.g., Questionnaire U.S.-L- #1, #2, & #4)	
B is higher at the starting point.(3)	e.g., "[Students might be thinking] B is higher at the starting point on the graph." (e.g., Questionnaire U.S.-L-#6)	
The points on the	e.g., "[They might be thinking] the	

graph is the speed.(1)	points on the graph represent the speed." (Questionnaire U.S.-L-#10)	
Do not know what speed means.(2)	e.g., "[They] might not understand what speed means." (Questionnaire U.S.-L-#7 & #11)	
	Correcting approaches	
Provide real-life examples to understand the relationship between distance, time and speed.(2)	e.g., "Provide real-life examples for students to understand the relationship between distance, time and speed." (Questionnaire U.S.-L-#2 & #3)	Improve understanding and memory of the relationship between distance, time and speed. (See interview U.S.-L-#3)
Teach the relationship between speed, time, and distance and initiate a discussion on steepness (slope) and height of lines (starting point) and how to determine these.(5)	e.g., "[I would] teach them the relationship between speed, time and distance. Ask students to discuss the steepness (slope) and the height of lines (starting point) of lines. Discuss how to determine the steepness and height." (Questionnaire U.S.-L-#1)	The comparison becomes easy if students understand the idea of steepness and height. (See Interview U.S.-L- #1)
Ask students discuss the concept of slope/rate of change.(4)	e.g., "[I would] ask students to define what rate of change or slope is. Then we would discuss what slope means on a graph." (Questionnaire U.S.-L-#4)	This problem is checking students' understanding of slope or rate of change. (See Interview U.S.-L-#4)

Carefully read the graph again to understand the information given in the graph.(1)	e.g., "I will ask my students to carefully read the graph again. I will ask them to write down the information given in the graph. I will then ask them to tell me what we can get from the information and make the comparison." (Questionnaire U.S.-L-#8)

Note. Numbers in parentheses indicate the number of responses that fall into that category.

There are five mathematical ideas/concepts that teachers in this group think are important for students to know in order to correctly solve the problem: meaning of x-value on the graph; slope/rate of change in graph and meaning; y-intercept; comparison of numbers and graphs; and meaning of distance, speed and time and the relationship between them.

Nine out of the twelve teachers in this group pointed out the importance of understanding slope/rate of change in graphs. For example, Ms. Fenning wrote in her questionnaire response, "In a uniform motion, the speed is the slope of each line. Comparing speed is comparing slope actually." (Questionnaire U.S.-L-#1)

Four out of twelve teachers in this group pointed to "*y*-intercept" to correctly answer the question in the scenario. For example, Ms. Haley stated in her questionnaire response, "Pay attention to the *y*-intercept and the relationship between the starting point and *y*-intercept. They are actually the same thing even they are in different forms." (Questionnaire U.S.-L-#3)

Two teachers mentioned that students need to know the meaning of distance, speed and time and how they relate to each other. For example, Mr. Jordon wrote in her questionnaire,

> [They need to know] the meaning of distance, speed and time and their relationship. Plus, they need to find the information on the graph. For example, do they know the *x*-axis indicates the time? Do they know the *y*-axis indicates the distance the object moves?" (Questionnaire U.S.-L-#6)

Additionally, two teachers mentioned in their questionnaire responses that "[students need to know] the meaning of *x*-values on the graph," (e.g., Questionnaire U.S.-L-#11) and two teachers wrote in their questionnaire responses that "[students need to understand] the comparisons of numbers of graphs." (e.g., Questionnaire U.S.-L-#5)

Five types of responses were found in this teacher group when asked what they think their students might be thinking when solving this problem. Two teachers wrote in their questionnaire responses that, "[Students might be thinking] B is higher at t=2 on the graph which means B moves faster." (Questionnaire U.S.-L-#3 & #8)

Six of the teachers wrote in their questionnaire responses that "[students might be thinking] B is higher on the graph." (e.g., Questionnaire U.S.-L-#1, #2, & #4) Three teachers wrote in their questionnaire responses that "[students might be thinking] B is higher at the starting point on the graph." (e.g., Questionnaire U.S.-L-#6)

One teacher wrote in her questionnaire response, "[They might be thinking] the points on the graph represent the speed." (Questionnaire U.S.-L-#10)

Two teachers wrote in their questionnaire responses that "[they] might not understand what speed means." (Questionnaire U.S.-L-#7 & #11)

When asked why they thought their students may have this thinking, two types of explanations were identified from interview data. Ms. Fenning and Ms. Iverson pointed to students' lack of math skills in comparing graphs. For example, Ms. Iverson explained in her interview,

> It's about comparison. They have difficulties in comparing stuff, graphs, functions…This problem is actually asking them to compare slope…but I doubt if they [will] figure this out if given this question. They will have to learn how to compare. This is a very important skill in math and it connects to their daily life. For example, they may have to compare cell phone plans, or they may have to compare trip routes, etc. This is what they are lacking of, I think. (Interview U.S.-L-#4)

The other four teachers interviewed in this group suggested that students may not fully know how to read the graph. For example, Ms. Haley said in her interview,

> Why do they think B is higher so B is moving faster? They do not understand the graph. What does the *x*-axis represent? What does the *y*-axis represent? Where can you figure out the speed from the graph? If they know how to read the graphs, they will not make this mistake. They have seen a lot of graph on the coordinate plane. But I doubt if they really pay attention to the *x*-axis and *y*-axis. They are kinda confused when the axis and the graph are given the 'real' meaning. They cannot make the connection… (Interview U.S.-L-#3)

When asked how they would help their students correct this mistake, teachers in this group came up with the following approaches based on their responses to the questionnaire. Two teachers, Ms. Gerold and Ms. Haley, wrote in their questionnaire

responses that they would "provide real-life examples for students to understand the relationship between distance, time and speed (*Questionnaire U.S.-L-#2, #3*). Their explanations for this choice were quite similar. For example, Ms. Haley explained in her interview,

> I would like to ask two students to come to the front to present a similar scenario. I would ask them to start two different points and to walk at different paces. I would ask other kids to record their time and distance, make a table of values and graph them…I think it is very visual. My kids like this hands-on activity… They feel they are part of the graph. They will understand what the graph really means…next time when they see this type of problem, the activity we do today will jump into their minds…they will remember the relationship between speed, time and distance and their representations on the graph… (Interview U.S.-L-#3)

Five teachers mentioned in their questionnaire responses that they would initiate discussion on steepness and height of lines. For example, Ms. Fenning wrote in her questionnaire response,

> [I would] teach them the relationship between speed, time and distance. Ask students to discuss the steepness (slope) and the height of lines (starting point) of lines. Discuss how to determine the steepness and height. (Questionnaire U.S.-L-#1)

She explained in her interview,

> As I said, this is about comparison. A discussion on steepness and height of graphs will help them understand what they are comparing and what the differences are if they use different 'parameter'. Once they grasp the idea of

steepness and height and what determines these, this comparison becomes fairly easy. For example, I ask which one is at a further position at a particular time point, they know it is a comparison of y-value; or if I ask which one is moving faster, they know it is a comparison of slope of a line or the coefficient of the linear term… (Interview U.S.-L- #1)

Mr. Jordon wrote in his questionnaire response, "[I would] initiate a discussion on steepness and height of lines in class. Discuss what (slope) determines the steepness of a line and what (y-intercept) determines the starting point of a line." (Questionnaire U.S.-L-#6)

He explained in his interview,

I think there is a lack of understanding of distance, time and speed relationship. It is an opportunity to teach them the relationship and their graphs. They will encounter a lot of problems like this. If they do not have a systematic knowledge of motions, uh, knowledge of reading graphs as well, it's gonna be hard for my kids to go through. (Interview U.S.-L-#5)

Four teachers mentioned in their questionnaire responses that they would ask students to discuss the concept of slope. For example, Ms. Iverson wrote, "[I would] ask students to define what rate of change or slope is. Then we would discuss what slope means on a graph." (Questionnaire U.S.-L-#4)

She said in her interview,

This problem is checking students' understanding of slope or rate of change. For middle school kids, we do not require them to use the vocabulary 'slope', but they know it. If they had solid understanding of slope, they would not make this type

of mistake… I will ask them to discuss the concept of slope here. What does a slope mean? How can you determine the value of a slope from an equation? If a line is very steep, is the slope big or small? I want them to make this connection using slope. They are going to use it a lot in linear functions. (Interview U.S.-L-#4)

Additionally, one teacher wrote in her questionnaire response that,

I will ask my students to carefully read the graph again. I will ask them to write down the information given in the graph. I will then ask them to tell me what we can get from the information and make the comparison. (Questionnaire U.S.-L-#8)

The approaches teachers in this group suggested for correcting this mistake were then categorized into four types. Type 1: Provide real-life examples for students to understand the relationship between distance, time and speed. Type 2: Initiate a discussion on steepness and height of lines and how to determine them. Type 3: Ask students to discuss the concept of slope/rate of change for graphs. Type 4: Ask students to carefully read the graph again to understand the information given in the graph.

Curricular knowledge.

In this section, I provide a description of the U.S. lower SES middle school math teachers' curricular knowledge (see Table 15). Specifically, I first present these teachers' responses to what instructional materials, textbooks, in particular, they use for teaching the topic of functions. The data source for this is the questionnaires collected from these teachers. I then present the teachers' explanations of how they use these instructional materials, textbooks, in particular, in the classrooms. The data source for this is the interviews. Lastly, I present these teachers' lateral and vertical curriculum knowledge. The data source for both the lateral and vertical curriculum knowledge is the interviews.

Table 15

U.S. Lower SES Teacher Group's Curricular Knowledge

Instructional materials	Standards (6)
	Connected Mathematics Project (7), *Glencoe Algebra* (3), *McDougal Littell Algebra* (3), and *Holt Mathematics* (2), etc.
	Teacher created materials (3) and outside materials (5).
The use of textbooks	Use a few examples/problems.
Functions and other disciplines	Include physics concepts in class.
	Little interdisciplinary work.
Functions and other topics in math.	Pattern, relation and function are connected

Note. Numbers in parentheses indicate the number of responses.

More than half of the teachers in this group stated in their questionnaire responses that they used standards as one instructional material (see e.g., Questionnaire U.S.-L-#1, #3, & #8). The frequently mentioned textbooks used to teach the topic of functions found in teachers' questionnaire responses were *Connected Mathematics Project*, *Glencoe Algebra*, *McDougal Littell Algebra*, and *Holt Mathematics*. Some teachers used one single textbook while other teachers used multiple textbooks in their respective classes. Three teachers mentioned that they used their own material for instruction (see Questionnaire U.S.-L-#4 & #6). Additionally, about half of the teachers in this group mentioned that they used outside materials to supplement textbooks in class (see Questionnaire U.S.-L-#1 & #3).

Based on the interview data, when asked about their use of textbooks, teachers suggested that they did use some examples or problems from the textbooks, but provided a lot of more examples or from their own resources. For example, Mr. Jordon said in his interview,

> We are teaching CMP (Connected Mathematics Project) in our school. But we feel very uncomfortable with it because we now do not feel it [is] presenting the most appropriate way of teaching some topics. So we create our own lessons, our own tasks, problems for kids here…we (teachers) do it together…I create everything. I'd say 90 percent of that is created of the homework assigned to my kids." (Interview U.S.-L-#5)

Ms. Haley, who did not use a textbook for instruction, explained to me in the interview,

> My kids do not have a book…See this journal? This is their book. Every kid has one journal…They're doing their notes all like this. Because they don't have a book, this is what they use and refer to throughout class. We were working on the-actually it was one of my seventh graders today. 'Ms Haley, which one's slow? M or B?' I said, "Well, it should be in your notes.' Pulled her notebook out, 'Oh, Okay. I got it.' They just go back to it and they know. It's like this was one of the – this was their point slope formula…They have the same way in every single lesson for the notes. It must have the essential question. The rules or formulas. The examples, problem. They have it in the journal. I also included problems similar to test items on AIMS... (Interview U.S.-L-#3)

According to teachers' interviews, when asked about their interdisciplinary work between functions (math) and other disciplines, teachers in the group provided the following responses. Ms. Fenning, Ms. Haley and Ms. Iverson explained that they incorporated some physics concepts when they teach functions (math). For example, Ms. Fenning said in her interview,

> It's because of this that I might actually do- especially when I get in here and talk about the exponential functions, I might talk about some of the physics ideas that they learned in the physics classroom…When I'm teaching functions, then I want to collaborate with the science teacher and put it all together in one great big unit. While she's teaching physics, I'm gonna be teaching linear systems. I'm gonna be teaching functions. They're gonna understand the mathematics that goes along with the science… (Interview U.S.-L-#1)

Differently, Ms. Gerold, Mr. Jordon and Ms. Kean pointed out that there was gap in the interdisciplinary collaboration. For example, Mr. Jordon said in his interview,

> …In elementary school, of course, you don't do it with different teachers, but you do tend to combine the two ideas, and teach a unit on something, which would include all the discipline. But you have to really make a conscious decision in middle school to do it…If we had professional learning communities with interdisciplinary stuff or did projects as a group could help bridge the gap. Yeah. But right now we do more collaboration with teachers from the same discipline rather than with teachers from other disciplines…. (Interview U.S.-L-#5)

When asked to describe the connections between functions and related knowledge in math learning, teachers explained that patterns, relations and functions are connected

and knowledge related to these build a large system, according to their interviews. For example, Ms. Gerold said,

> …Patterns, relations and functions. They are a large system. These three are always in this order in most books. So they can see functions in this system. Kids can see relationships because functions, I mean, they explain everything… We teach these lessons and connect them to the units. These units form the system. All the topics or concepts involved in this system are connected… (Interview U.S.-L-#2)

Similarly, Ms. Kean said in her interview,

> Equations, relationships, functions…they all build together. Right. So it's not as- it's like you have this prior knowledge and you scaffold onto it. They've already had to deal with expressions, equations, dealing with variables, X, Y, Z, Q whatever they are. Then you start to look at that in terms of the coordinate plane. Then you start to graph a linear equation. Then from there we talk about well what is a function… These are all related to function. They are building together… (Interview U.S.-L-#6)

The Chinese Higher SES Middle School Math Teacher Group

Teachers' instructional knowledge for introducing the topic of functions.

In this section, I provide three sub-sections to describe the Chinese higher SES middle school math teachers' instructional knowledge for introducing the topic of functions. These subsections are the goal of teaching introductory class of functions, the construction of mathematical tasks and the use of representations of functions. In each subsection, a description of teachers' instructional decisions is presented based on the

data collected from lesson plans, followed by teachers' explanations of their instructional decisions from semi-structured interview data. In total, eleven introductory lesson plans were collected from the Chinese higher SES middle school math teacher group and four of the teachers in this group were interviewed. All the quotes in this section were translated by the researcher.

The goals of teaching introductory class of functions.

In this study, the Chinese higher SES middle school math teachers' goals of teaching introductory class of functions covered ten aspects. They expected their students to be able to understand the relationship between pattern, relation and function; that function is one input with exactly one output; the concept of input/output; domain and range; change and correspondence; constant and variable; range of *x*; slope or rate of change; linear functions; independent and dependent variable; algebraic functions; and three forms of representation of function. See Table 16.

Table 16

Instructional Goals for the Chinese Higher SES Teacher Group

Goal	Number of response	Example	Teachers' explanations
Pattern, relation and function.	5	e.g., "Students will be able to understand the relationships between patterns, relations and functions." (Lesson plan Chinese-H-#2)	Math is connected; most difficult part is finding the pattern.
Function as one input with exactly one	6	e.g., "[Students are expected to] understand the definition of function. A function is a rule that	Standards and textbook; most difficult part is

output.		assigns to each input (independent variable) exactly one output (dependent variable).” (Lesson plan Chinese-H-#3)	back to the definition.
Change and correspondence.	3	e.g., “Understand what change means and what correspondence means.” (Lesson plan Chinese-H-#7)	
Constant and variable.	5	e.g., “(They will) understand what a constant is and what a variable is in this class.” (Lesson plan Chinese-H-#4)	Textbook/sequence the curriculum.
Range of x-value and range of y-value or domain and range.	6	e.g., “They will be able to identify the domain and range of a function.” (Lesson plan Chinese-H-#5)	Make sense of a function; high school learning concern.
Slope/rate of change	1	e.g., “Students will understand the definition of rate of change (slope).” (Lesson plan Chinese-H-#9)	
Linear functions	1	e.g., “Students will be able to identify linear functions.” (Lesson plan Chinese-H-#10)	
Independent and dependent variable.	3	e.g., “Understand what independent variable and dependent variable represent in a function.” (Lesson plan Chinese-	

		H-#6)	
Algebraic functions.	8	e.g., "Students will be able to understand and write the algebraic form of a function." (Lesson plan Chinese-H-#6)	The most important form to understand a function; high-school exam.
Three forms of representations.	7	e.g., "(Students will) understand that a function usually have three forms of representations - equation, tables and graphs." (Lesson plan Chinese-H-#9)	Standards and textbooks; and understand a function thoroughly.

In the eleven introductory lesson plans collected from this group, six of them mentioned that knowing the definition of function as one input having exactly one output is an important goal for the introductory lesson. For example, one teacher wrote in his lesson plan, "[Students are expected to] understand the definition of function. A function is a rule that assigns to each input (independent variable) exactly one output (dependent variable)." (Lesson plan Chinese-H-#3)

When asked why this was an important goal for this introductory lesson, all the teachers interviewed mentioned the *New Standards* and textbook. For example, Ms. Zhao explained,

> We follow the standards and textbook…I think there must be some important reasons in our standards and textbook to put function definition as a goal in the introductory classes… To learn a new topic, a new concept, I think it is important to know what the definition is, right? I think understanding the definition is the

first step to understand a new concept and the related math problem. We do it for every new topic. (Interview Chinese-H-#1)

Similarly, Mr. Qian also explained,

…It is important to put definition in the first class of learning a new concept. Our standards and textbook have a guide for this. It is important to sequence the curriculum, I think. The curriculum must have taken students' psychological development at this age into account and it makes sense to them to know the definition first. (Interview Chinese-H-#2)

He also added,

In my opinion, understanding the concept is actually the most difficult part of learning a topic. Quite often, we neglect the learning of the concept and the definition in teaching…we always think of solving the problem, the procedures, or getting an answer, as we move on with a topic. The definition is something that we leave behind…Students can solve a problem without understanding the concept. But that's not a real understanding. I know it is difficult for them to 'really' understand a concept at the beginning. Actually I think usually students have a better understanding after they see many examples, functions, equations, graphs. And when they look back, they will find it is easier to understand the concept. That's why we review all the time. We try to remind them all the time. We try to connect everything along the way for them. But it does not mean it is not important to be aware of the concept, the definition at the very beginning. I will spend a lot time explaining this concept. I hope they will pay attention to it… (Interview Chinese-H-#2)

Five of the eleven teachers included understanding constant and variable as a goal in the introductory class. Ms. Sun highlighted this goal in her lesson plan, "[They will] understand what a constant is and what a variable is in this class." (Lesson plan Chinese-H-#4)

She explained this to me in the interview,

> I do not think it is a difficult concept for my kids to understand actually. I do not need to spend a lot of time on it…If you look at our textbook, the Renjiaoban, Constant and Variable is the first section in this chapter. Yep, we follow the textbook but we do not spend much time on it. We move to functions quickly. (Interview Chinese-H-#3)

Ms. Li had a similar explanation that, "*Constant and Variable* is the first section in the textbook. I also address this concept in my class. I just do not spend much time teaching this. It is quite easy for my kids." (Interview Chinese-H-#4)

Five teachers mentioned in their lesson plans that understanding patterns, relations and functions was an important goal in their introductory class. For example, Mr. Qian wrote in his lesson plan that, "Students will be able to understand the relationships between patterns, relations and functions." (Lesson plan Chinese-H-#2)

When asked why he set this goal, Mr. Qian explained in the interview,

> Patterns, relations, and functions…Seems different but they have all kinds of relationships and connections… I want my kids to know math is connected. What they learn today can be related to what we learn someday in the past and can be related to what they will learn someday in the future. I need them to build that network…For functions, I want my kids to know how they are similar to relations

or patterns and how they are different. Is a function a relation? Is a relation a function? I want them to know that many relationships in math are directional, uh, or cannot be reversed. (Interview Chinese-H-#2)

A different explanation was found in Ms. Li's response,

Yes, we move from patterns and relations to functions. But in my opinion, finding patterns or relations can be very difficult. There is one question item asking students to find the pattern on the high school entrance exam almost every year. A lot of my students fail on this question even if they are smart… (Interview Chinese-H-#4)

Understanding "range of *x*-value", "range of independent variable" "range of *y*-value" and "range of dependent variable, or "domain and range" was another important goal that six of the teachers in this group explicitly stated in their lesson plans. For example, Ms. Zhao wrote in her lesson plan that, "Students are expected to understand what domain and range mean." (Lesson plan Chinese-H-#1)

Ms. Li wrote, "They will be able to identify the domain and range of a function." (Lesson plan Chinese-H-#5)

When asked why this was an important goal for them, Ms. Zhao and Ms. Li provided similar explanations. For example, Ms. Zhao said,

The range of x or domain as we say in high school, is a very important concept, I think. Kids tend to neglect this, however. Without the range of x, you cannot make sense of a function. What values can this x be? Should be integers? Or continuous values…Should it be negatives? For example, it will not make any sense when you put negatives in the x which represents for gas in a car. Right? I

want them to be aware of this. Moreover, they are going to deal with domain and range seriously in high school. I want them to get this now. It is gonna be very hard for them to change if they get the misconception that they do not need to pay attention to the domain. (Interview Chinese-H-#1)

Ms. Li said,

I told them to be careful about the values of x. a function cannot make sense in real-life situations without the values of x. It is simple. Like, you cannot have half of a person when you count your classmates. You have to make sense. The values of x determine the values of y. I told them that they will learn it again in high school. You will learn something called domain. They are going to learn much more complex functions and domain is going to play an even more important role. I want them to have this sense now. Do not wait until high school. I train them to do so from the very beginning. (Interview Chinese-H-#4)

Grasping representations of functions as one of the important goals was explicitly stated by most of the teachers in this group. Specifically, seven teachers mentioned that it is important for students to be able to understand multiple representations of functions – equations, tables and graphs – in the introductory class. For example, one teacher wrote in her lesson plan, "[Students will] understand that a function usually have three forms of representations –equation, tables and graphs." (Lesson plan Chinese-H-#9)

Moreover, eight teachers pointed out the importance of understanding algebraic functions. For example, one teacher wrote in his lesson plan that, "Students will be able to understand and write the algebraic form of a function." (Lesson plan Chinese-H-#6)

When asked to explain the goal of understanding multiple representations, two reasons were identified from teachers' interview data. One pointed to the standards and textbook and the other pointed to the thorough understanding of a function. For example, as Ms. Sun explained in her interview,

> Students are required to understand multiple representations of functions based on the New Standards. More exactly, three forms of representations in our textbook-equations, tables, and graphs. Students are expected to understand what a function is from all these forms. I think it is important for them to know all these forms because it definitely will broaden their horizon. Not just limited in one form. (Interview Chinese-H-#3)

Similarly, Ms. Li explained in the interview,

> Our standards and textbook ask them to understand three forms of representation--equation, table and graph. Kids will get a more thorough understanding of the concept of function through different forms of representation. I encourage them to use multiple representations from the very beginning. (Interview Chinese-H-#4)

Ms. Li, when asked why she also included understanding algebraic functions as a separate goal in her lesson plan, explained to me,

> Algebraic function is still the most important form of representations. You can figure out all the features of a function through its algebraic form. The most important way to understand a function is through its equation, seriously… It can be very difficult. It (algebraic function) frequently 'shows up' on the high school entrance exam. Sometimes it (algebraic function) 'shows up' alone. Sometime it

'shows up' together with function graph… I need them to pay more attention to it. (Interview Chinese-H-#4)

Similarly, Ms. Zhao explained in her interview,

> I include this goal in my introductory class basically because in my opinion, to be able to understand and apply equations is the most important task that my kids will be ask to complete in the future. Equation is important to understand a function…it is the foundation. Without understanding the equations, we cannot really understand a function, not to mention solving problems related to functions. And also, I see using algebraic functions as reflecting students' ability of abstract thinking… (Interview Chinese-H-#1)

Three teachers wrote in their lesson plans that students were expected to understand change and correspondence. For example, one teacher wrote, "Understand what change means and what correspondence means to better understand a function" (Lesson plan Chinese-H-#7)

Similarly, another teacher wrote, "Understand functions from the perspective of change and correspondence." (Lesson plan Chinese –H-#4)

Three teachers mentioned in their lesson plans, "Students will understand what independent variable and dependent variable represent in a function." (Lesson plan Chinese-H-#6)

Additionally, only one teacher wrote, "Students will understand the definition of rate of change (slope)." (Lesson plan Chinese-H-#9)

Only one teacher wrote, "Students will be able to identify linear functions." (Lesson plan Chinese-H-#10)

The construction of mathematical tasks.

Table 17

Mathematical Task by Cognitive Demand for the Chinese Higher SES Teacher Group

Cognitive demand of mathematical task	Example	Teachers' explanations
	Lower-level demand	
Memorization.(0)	N/A	
Procedures without connection to concepts/understanding/meaning.(1)	e.g., If $y=2x+1$, please fill in the following table. | x |				| | y |					(Lesson plan Chinese-H-#7)	
	Higher-level demand	
Procedures with connection to concepts/understanding/meaning.(22)	e.g., Xiaoli is going to a bookstore to buy some notebooks. The relationship between the price of a notebook (Q) and the quantity of the notebooks (x) is shown in the table. Could you please write an algebraic function for this relationship? Q=_____. | x/Book | 1 | 2 | 3 | 4 | … | | Q/RMB | 5 | 10 | 15 | 20 | … | (Lesson plan Chinese –H-#7)	New standards and high school entrance exam; a real understanding is not equal to know the procedures.
Doing	e.g., See below.	High school

mathematics.(4)	entrance exam; student ability is higher and they can do math.

Note. Numbers in parentheses indicate the number of responses that fall into that level.

Twenty-seven mathematical tasks were constructed in the introductory lesson plans of the Chinese higher SES middle school math teachers. Only one out of the twenty-seven mathematical tasks was constructed at the lower demand level – procedures without connections to meaning, concepts or understanding. For example,

If $y=2x+1$, please fill in the following table

x				
y				

(Lesson plan Chinese-H-#7)

Twenty-two of the twenty-seven mathematical tasks were constructed at the level of cognitive demand focused on procedures, with connections to concept, understanding or meaning. Examples of these mathematical tasks are as follows,

The rate of gas is 6 RMB/L. Please fill in the table

x(L)	1	2	3	4	5	…
y (RMB)						…

What is the relationship between the quantity of gas you add and the amount of money you pay for gas? Can you write an equation for it? Is there a range for the quantity of gas (x) you add? Why? (Lesson plan Chinese-H-#1)

The length of a side of a square is marked as *a*. Fill in the table. How does the area of a square change?

Length of a side (*a*)	1	2	4	6	9	…
Square						

Does the area of a square have a particular value when the length of a side has a particular value? Is this a function? Why? (Lesson plan Chinese-H-#3)

Xiaoli is going to a bookstore to buy some notebooks. The relationship between the price of a notebook (Q) and the quantity of the notebooks (*X*) is shown in the table. Could you please write an algebraic function for this relationship?

Q=_____.

X/Book	1	2	3	4	…
Q/RMB	5	10	15	20	…

(Lesson plan Chinese-H-#7)

The formula of calculating the volume of a sphere is $V=\frac{4}{3}\pi R^3$. *V* represents the volume and *R* is the radius. Please determine _____is (are) the constant and _____is (are) the variable(s). (Lesson plan Chinese-H-#10)

All the four teachers interviewed pointed to the new math standards and high school entrance examination as important reasons for constructing mathematical tasks connecting to concept, understanding, or meaning. Ms. Zhao explained in her interview,

The new math standards actually decrease the complexity of calculation…it's focused on students' ability of solving problems…how you can understand and analyze a problem….Basically now you will not see questions merely focusing on calculation itself on the high school entrance exam. Most of the questions need kids to show their understanding of the meaning. Read and understand the problem, then analyze and solve it… (Interview Chinese-H-#1)

Mr. Qian said,

Under the new math standards, we have been transitioning to a new stage of learning and teaching, I think. Old testing emphasizes a lot procedural stuff. Sometimes, students simply memorized things. But now it does not work. For example, if you look at the question items on the high school entrance exam, you will find a lot of questions asking students to think, to understand the meaning. Some even ask them to model a problem mathematically, that is, they have to read a long paragraph of words which seems not that related to math and to find the connections between what they learned in math class and the problem… (Interview Chinese-H-#2)

Ms. Sun said,

[There are] big changes in the new standards and new high school entrance exam... For function, usually we did not do that good on Question 8, 12, 22, 23, 24 and 26 on high school entrance exam. For example, Question 8 asks students to show their understanding of function graphs. Students need to be familiar with function graphs and their meaning in real-life situations… (Interview Chinese-H-#3)

Ms. Li explained in her interview,

> The standards keep making some small changes every year these years. High school entrance exam keeps changing accordingly. But one point that has not been changed in the past few years is that it tests students' ability of understanding the meaning of a problem and apply their knowledge learned from class into a new context. Students sometimes cannot make connections between a new situation and what they already knew… (Interview Chinese –H-#4)

Teachers also highlighted the intrinsic value of understanding the meaning behind the concepts. According to Mr. Qian,

> I want them to know that math is not something that has nothing to do with their real life… It actually comes from our lives. Understand the concept. Understand the meaning under each concept. Understand their connections to our lives. I think this is important, really important for our kids…even if it may not appear on the test. (Interview Chinese-H-#2)

Ms. Li also pointed out the importance of making meaning of math learning. She said,

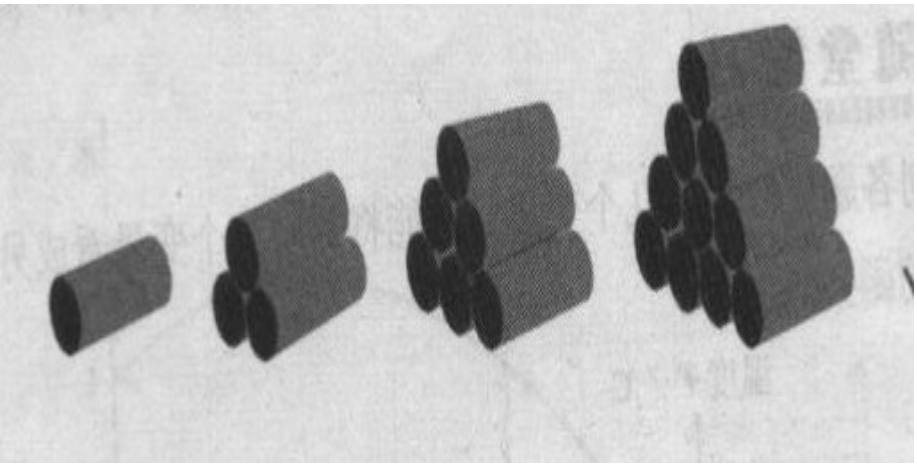
> The first class of a topic is important. I want my kids to know the concept… uh, know the meaning from the very beginning. If I emphasize the procedures at the first class, they will think it is not important to understand the concept. They just want to know how to get y and x values, etc… (Interview Chinese-H-#4)

Four mathematical tasks were constructed at the "doing-mathematics" level. An example of this type of mathematical tasks in an introductory class of functions is as follows,

Observe the first graph. What do you think the variables could be in this graph? Could it be the layers and the quantity of cylinders? How do they change together? What is the relationship between them?

Then observe the second graph. What do you think the variable could be in this graph? Could it be the time of walking and the distance of walking for a turtle? How do they change together? What is the relationship between them?

What do you think the similarities/differences between the two relationships indicated in these two graphs? What do you find? Be prepared to present to your group members.



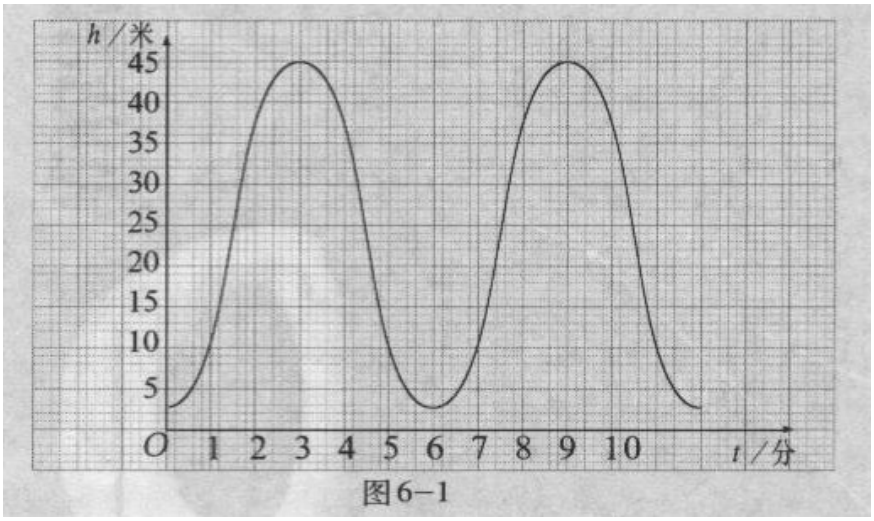


图6—1

(Lesson plan Chinese-H-#3)

When asked why they constructed "doing-mathematics" type of mathematical tasks, Mr. Qian and Ms. Sun responded as follows.

Mr. Qian said in his interview,

> I know this is difficult for them, especially at the introductory class. But I have my belief in my kids. They are smart. They like math… read and understand graphs. Find patterns. Make comparisons. All these skills are important and hard to grasp. But they need to grasp. A similar question will appear on the high school entrance exam. That is Question 24. This question asks kids to make connections between knowledge points related to functions, combine equations and graphs, and make predictions. A very comprehensive problem… The correctness rate is low on this item every year. But I know some of my kids. They get it. I train them from the very beginning. We practice on similar questions. (Interview Chinese-H-#2)

Ms. Sun said in her interview,

I always give my kids this type of task to solve as an extension or application. They have the ability to solve this task. I always wanna see how far they can go... You have to push the boundary... My kids are at a higher level. I know they can. Test items on functions can be very difficult on the high school entrance exam. One of the last three questions is a comprehensive question on functions. I want them to practice from the start… I believe in them even it's difficult. (Interview Chinese-H-#3)

The use of representations of functions.

Teachers in this group used four forms of representations of functions in their lesson plans. These forms of representations were equations or algebraic form of functions, graphs, tables and verbal. Some examples for each form of representations in this group are as follows.

Equations:

For example,

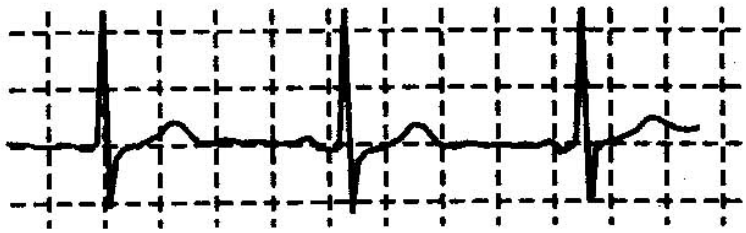
A person drives a car from Rugao to Nanjing. The distance between these two cities is 250 kilometers. The relationship between the speed (v: kilometer/hour) and the driving time (t: hour) is $v = 250/t$.

In this changing process, the distance is the constant; and the speed (v) and the time (t) are the variables. (Lesson plan Chinese-H-#5)

Graphs:

For example,

This is the electrocardiogram for a person. The *x*-axis represents the time. The *y*-axis represents the biological electric current. In this graph, is there only one *y*-value for every *x*-value? Is this a function?



(Lesson plan Chinese-H-#2)

Tables:

For example,

The length of a side of a square is marked as a. Fill in the table. How does the area of a square change?

Length of a side (*a*)	1	2	4	6	9	…
Square						

Does the area of a square have a particular value when the length of a side has a particular value? Is this a function? Why? (Lesson plan Chinese-H-#3)

Verbal:

For example,

Share with your shoulder partners three real-life examples of functions. Explain to her or him why they are functions. (Lesson plan Chinese-H-#4)

How did teachers in this groups use these forms of representations in their introductory lesson plans? As shown in Table 4.15, among the eleven teachers whose

lesson plans were collected in this group, all of them used algebraic representations of functions throughout their lesson plans. Three of these teachers used only equations in their lesson plans. Three used both equations and tables in their lesson plans. Four of the teachers used equations, tables and graphs in their lesson plans. One teacher used tables, graphs, equations and verbal representations in her lesson plan.

Table 18

The Use of Representations by the Chinese Higher SES Teacher Group

Form of representation	Number of response
Equation only	3
Equation and table	3
Equation, table and graph	4
Table, graph, equation, and verbal	1

Teachers' explanations:

Equation, table, and graph are three forms of representations of functions.

Equation is the most important form of representation as determining the nature of a function; appearing in high school entrance exam; connecting knowledge points.

Understand translation between equation and graph.

When asked to explain their attitudes toward the use of representations of functions in the interviews, teachers provided the following explanations.

First, they expected students from the first lesson to know the three forms of representations of functions as required by the standards (i.e. equations, tables, and graphs). As Ms. Zhao explained in her interview, "[Equations, tables and graphs] are the three forms that our standards ask students to understand." (Interview Chinese-H-#1)

She said,

When we talk about representations of function, we are actually talking about equations, graphs and tables. This is stated in our standards…and textbook too. Students should be able to deal with all the three forms. I expect them to do so. (Interview Chinese-H-#1)

She also added,

I think it might be better to add verbal representation too. I want my kids to use their words to describe a function, to come up with stories, to connect to their real-life situations…If they really understand a function, I expect them to use verbal representations. That is real understanding to me." (Interview Chinese-H-#1)

Similarly, Mr. Qian said in his interview,

…My kids need to understand symbolic, tabular and graphical representation of functions. It is explicitly stated in the standards. You know standards are our flags which we follow closely. For a long time, they will play with these representations. Each of them has a special role in understanding a function. If you want to understand a function thoroughly, you need to know all the three forms. (Interview Chinese-H-#2)

Second, of the different forms of representations, equation was highlighted as the most important. Ms. Zhao explained in her interview,

Equation is the most important representation that I require my kids to understand…it is the foundation… we can uncover all the features of a function through analyzing its equation representation. For my class, I always do the

equation first for linear, proportional, inverse-proportional and quadratic…Once they understand the equations, we then move to visualize it using graphs… (Interview Chinese-H-#1)

Mr. Qian said in his interview,

…algebraic expression of functions is the most important one…I see algebraic expression as similar to equation we have learned. I mean think about its role in the kids' math learning. We move from numbers to equation, we move from equation to function…the concept of equation is so important… from concrete to abstract… Equation stands in the center of our learning of math, I think. You can understand a function completely using its equation… (Interview Chinese-H-#2)

As Ms. Sun explained in her interview, she had an additional interest in ensuring her students' success in the entrance exam. She said,

I think equations are the most important and difficult representation so that I want my kids to spend time to practice. It is frequently tested on the high school entrance exam. The most difficult item usually tests students' understanding of the equation and its graph. (Interview Chinese-H-#3)

Ms. Li explained her attitude to me in the interview,

Equation is important. I think my kids can play with symbols…or equations. You know you do not need to use manipulatives or use graphs to help them understand function. You can use equation to show them what a function is, where functions come. Everything is building upon equations. That's what algebra is about… (Interview Chinese-H-#4)

Third, teacher emphasized the importance of learning to transfer between equations and other forms of representations, especially between equation and graph. For example, Mr. Qian explained in the interview,

> We usually ask students to solve problems on equation and its graph…this type of problems are the most difficult one appeared on the high school entrance exam. I think the transfer between these two representations is key to solving this type of problems… students need to be able to write an equation based on its graph, and they are also expected to draw a graph based on the equation. But you know this is not that difficult for my kids if the information included isn't enough. The most difficult part is that the information given in the graph is incomplete. They need to read the scenario to organize information and then combine any information in the graph in order to get an equation…Anyway, it is important to understand how to transfer between equations and graphs of functions. (Interview Chinese-H-#2)

Similarly, Ms. Li said in her interview,

> I think it is important to know how to transfer between equation and other forms of representation especially between equation and graph. Equation and graph together determine the nature of a function and what it looks like. To solve a problem of functions usually requires students' use of an equation and its graph. This is difficult for kids. I want them to practice… (Interview Chinese-H-#4)

Teachers' knowledge of student understanding of functions.

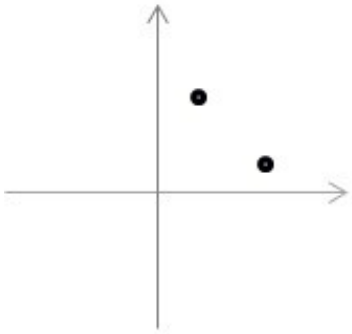
In this section, I provide detailed description of teachers' responses to two scenarios of students' mistakes. The first scenario is about students' mistakes on drawing function graphs through two given points on a coordinate plane. The second scenario is

about students' mistakes in comparing two linear functional graphs on a coordinate plane. For each scenario, I describe teachers' knowledge of students' understanding of functions from three aspects, based on their responses in the questionnaire. First of all, I describe the mathematical ideas that teachers think are important for students to correctly solve the math problem; second, I describe the "thinking" these teachers suggest their students might have leading to the mistake; and third, I describe teachers' approaches to correcting their students' mistakes. In each of the aspects, a description of teachers' responses is presented based on the data collected from questionnaires, followed by teachers' explanations collected from their semi-structured interviews. In total, fourteen questionnaires were collected from the Chinese higher SES middle school math teacher group and four of the teachers in this group were interviewed.

Scenario 1.

A review of the scenario:

If a student is asked to give an example of a graph of a function that passes through the points A and B, she gives an example as shown in Figure 2. When asked if there is another answer for this question, she says "No".



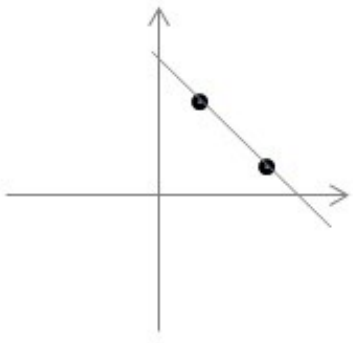


Table 19

Chinese Higher SES Teachers Dealing with Students' Mistake on Drawing Functional graph(s)

Teachers' response	Example	Teachers' explanations
Function definition: One input exactly with one output.(9)	e.g., "[Students need to] understand the definition, the concept of function." (Questionnaire Chinese-H-#3)	
Function graphs come in many shapes.(11)	e.g., "[They need to know that] not every function has a correspondent equation. Functions come in all kinds of shapes." (Questionnaire Chinese-H-#7)	
Linear-function features and non-linear graphs.(6)	e.g., "[They need to know] what a linear function is and what a non-linear function is. What is difference between linear and nonlinear functions?" (Questionnaire Chinese-H-#5)	
Equation determines the shape of a graph.(6)	e.g., "[They need to know] nonlinear functions. By the ninth grade, they learn inverse-proportional and quadratic functions. Be able to write equations for these functions. Be able to graph them on the coordinate plane based on the equations. Those equations determine what the graphs look like." (Questionnaire Chinese-	

	H-#7)	
Connecting two points create a line.(12)	e.g., "[Students would be thinking] connecting two points create a line on a coordinate plane." (e.g., Questionnaire Chinese-H-#1, #3, #5, & #7)	Fixed-thinking after learning linear functions (See Interview Chinese-H-#1 & #2)
A function is a line (a linear function).(2)	e.g., "[students would be thinking] functions are lines or linear functions." (Questionnaire Chinese - H-#8 & #11)	
Let students take the lead to discuss which graphs are functions using the definition of function.(3)	e.g., "[I would] let my kids take the lead. [I would] ask my students to draw a couple of graph by themselves. [I would] then discuss with the whole class which ones are functions and which one are not. Student would be expected to understand the definition of function." (Questionnaire Chinese-H-#5)	Kids are smart and they can take solve the problem by themselves.(See interview Chinese-H-#3)
Directly give students parabolas as counter-examples.(3)	e.g., "Give students a parabola." (e.g., Questionnaire Chinese-H-#9 & #12)	
Make up stories/construct scenarios and graphs (5)	e.g., "[I would] give them some real-life examples of functions including linear and nonlinear functions. [I would] ask them to tell me the differences between them."	Scenarios or real-life situations are more memorable than those equations or graphs in understanding what a

	(Questionnaire Chinese-H-#1)	function really means. (See Interview Chinese –H-#1& #2)
Use equation of non-linear functions the then graph them (3).	e.g., "[I would ask them to] write an inverse-proportional function and graph it on the coordinate plane. Write a quadratic function and graph it on the coordinate plane. Observe and tell the class what their differences from linear functions graphs. Do they share anything in common on the graph?" (Questionnaire Chinese-H-#7)	The nature of the algebraic form of a function determines the location and the shape of the functional graph. Equation is the basis. (See Interview Chinese-H-#4)

Note. Number in the parentheses indicates the number of responses that fall into that category.

There were five mathematical ideas or concepts mentioned in teachers' questionnaire responses in this group: function definition; non-linear functions and graphs; functions come in many shapes; linear-function features; and an equation determines the shape of a graph.

Eleven out of the fourteen teachers in this group mentioned that students need to know that functions come in many shapes for correctly answering the question in the scenario. As Ms. Li wrote in her questionnaire response, "[They need to know that] not every function has a correspondent equation. Functions come in all kinds of shapes." (Questionnaire Chinese-H-#7)

Nine of the teachers in this group pointed out in their questionnaire responses the importance of knowing function definition as every input having exactly one output. As

Mr. Qian wrote on his questionnaire, "[Students need to] understand the definition, the concept of function." (Questionnaire Chinese-H-#3)

Six teachers mentioned the importance of being familiar with linear function features and non-linear graphs. For example, one teacher wrote, "[They need to know] what a linear function is and what a non-linear function is. What is difference between linear and nonlinear functions?" (Questionnaire Chinese-H-#5)

Additionally, six of the teachers pointed out that knowing "equation determines the shape of a graph" was important for students to correctly solve this problem. As Ms. Li highlighted in her questionnaire response,

> [They need to know] nonlinear functions. By the ninth grade, they learn inverse-proportional and quadratic functions. Be able to write equations for these functions. Be able to graph them on the coordinate plane based on the equations. Those equations determine what the graphs look like. (Questionnaire Chinese-H-#7)

When asked what they thought their students would be thinking when solving this particular problem, teachers gave the following responses to the questionnaire. Two teachers wrote in their questionnaire responses that, "[students would be thinking] functions are lines or linear functions." (Questionnaire Chinese-H-#8 & #11)

The other twelve teachers wrote in their questionnaire responses that "[students would be thinking] connecting two points create a line on a coordinate plane." (e.g., *Questionnaire Chinese-H-#1, #3, #5, & #7*) All the four teachers interviewed in this group provided the response, mentioning that their students would be thinking "two points make a line". When asked why they thought their students would have this

thinking or misconception in the follow-up interviews, these teachers' explanations were quite similar – saying that it is a result of fixed-thinking after a long-time linear function learning. For example, Mr. Qian explained in his interview,

> It is a typical fixed-thinking. When we teach them linear function, we emphasize a lot on 'two points create a line' which mean if you are given two points on a coordinate plane, you can draw a line, or you can get a linear function. Once you jump out of the linear function area, uh, I mean, you are from a perspective of a function, not a linear function, but they are still in that linear mode. (Interview Chinese-H-#2)

Ms. Zhao also had a similar explanation in her interview,

> Kids get fixed-thinking easily. It is hard for them to jump out. You teach them something, they are in it. For example, I teach the ratio for two weeks, then all they remember in that two weeks is ratio. Same for linear functions…we say 'connecting two points create a line on a coordinate plane', they practice it a lot, all they can remember is 'connecting two points creates a line on a coordinate plane'. All the fixed-thinking… (Interview Chinese-H-#1)

She also added,

> I mean they still do not develop the ability of connecting knowledge points, building a network of math knowledge. Their knowledge is sliced…if they can look at the problem from a bigger perspective of function rather than lines or linear, they would not make this mistake. (Interview Chinese-H-#1)

When asked how they would help students correct this mistake in the scenario, teachers in this group provided the following approaches according to their responses to

the questionnaire. Three teachers mentioned that they would let students take the lead to discuss which graphs are functions using the definition of function. For example, Ms. Sun wrote in her questionnaire response,

> [I would] let my kids take the lead. [I would] ask my students to draw a couple of graph by themselves. [I would] then discuss with the whole class which ones are functions and which one are not. Student would be expected to understand the definition of function. (Questionnaire Chinese-H-#5)

She added in her interview,

> My kids are smart, seriously. They definitely can do it by themselves. I would ask them to draw a couple of graphs. These can be anything as long as they meet the criteria of being a function – one input with exactly one output. I would initiate a discussion among them. As more as they can draw, they will find out you can as infinite graphs as you can to connect two points in the graph. (Interview Chinese-H-#3)

Five teachers mentioned in their questionnaire responses that they would construct scenarios for linear and nonlinear function and graphs to help students correct the mistake. For example, Ms. Zhao wrote in her questionnaire response, "[I would] give them some real-life examples of functions including linear and nonlinear functions. [I would] ask them to tell me the differences between them." (Questionnaire Chinese-H-#1)

She explained in her interview,

> I think it [correcting this mistake] is a good opportunity to learn what a function is again, including linear and nonlinear functions, in contexts. I cannot simply give graphs or equations to them, saying this is the correct answer. I would like to ask

them to explore by themselves and this exploration can be related to some real-life situations…It gets back to what I mentioned 'making math learning meaningful'. Scenarios or real-life situations are more memorable than those equations or graphs. (Interview Chinese-H-#1)

Mr. Qian, who also mentioned in his questionnaire response that he would use real-life examples to help correct this mistake, explained in his interview,

I think understanding the concept of function is key to solving this problem. I would like to use some stories or examples from my kids to reteach the concept, the definition of function… They will understand what change and correspondence mean, what one input with exactly one output means, and what a function means. Then this problem will not be a problem for them…I mean once they see things from 'functions', a real understanding of functions, it's going to be difficult to connect two points or draw graphs. (Interview Chinese-H-#2)

He also added,

I think it is also important for my kids to know they do math under a certain conditions. They will learn to analyze those conditions based on the information given in the problem, which asks them to read, to understand the problem. This is an important math skill. Constructing scenarios is a way to help them read and understand information given in the situation… (Interview Chinese-H-#2)

Three teachers mentioned that they would use equations of non-linear function and their graphs to help students correct the mistake. For example, Ms. Li wrote in her questionnaire response,

> [I would ask them to] write an inverse-proportional function and graph it on the coordinate plane. Write a quadratic function and graph it on the coordinate plane. Observe and tell the class what their differences from linear functions graphs. Do they share anything in common on the graph? (Questionnaire Chinese-H-#7)

She explained to me in the interview,

> As I said, it is important to know the equations… They move from numbers, all specific numbers, to variables, equations, and algebraic functions…A big jump toward abstract thinking for math. At this stage, I would like to ask them to write a quadratic equation and graph it. All the features of quadratic functions can be found from the equations. For example, if the coefficient of the quadratic term is positive, the parabola extends upward, and if the constant term is negative, the y-intercept will be negative…the equation determines the location and the shape of the graph. Equation is the basis. Everything gets back to equations at this stage. (Interview Chinese-H-#4)

Three teachers mentioned in their questionnaire responses that they would "give students a parabola." (e.g., Questionnaire Chinese-H-#9 & #12)

Teachers' approaches to correcting students' mistake were then categorized into four types. Type 1: Let students take the lead to discuss which graphs are functions, using the definition of function. Type 2: Directly use parabolas as counter-examples. Type 3: Construct scenarios or create stories for linear and nonlinear functions and graph them. Type 4: Provide examples of nonlinear function equations and graph them on the coordinate plane.

Scenario 2.

A student is given the position vs. time graph as presented below. When asked to compare the speeds of the objects at time $t = 2$ sec., the student responds by saying that Object B is moving faster.

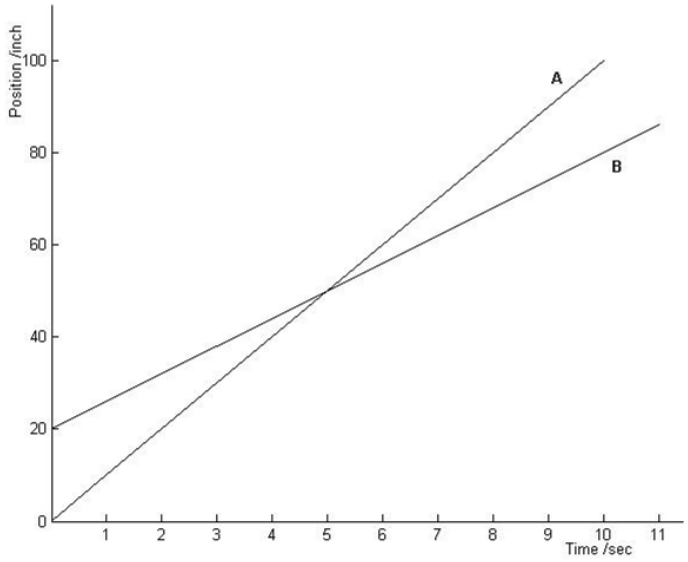


Table 20

Chinese Higher SES Teachers Dealing with Students' Mistake on Comparing Linear Functions on A Coordinate Plane

Teachers' response	Example	Teachers' explanations
	Mathematical ideas	
Understand the meaning of function graphs in real life.(2)	e.g., "[Students need to] understand the meaning of function graphs in real life situation." (Questionnaire Chinese-H-#9)	
Meaning of x-value on the	e.g., "[Students should] know the meaning of v-values on a graph."	

graph.(1)	(Questionnaire Chinese-H-#11)
Slope/rate of change in graph and meaning.(7)	e.g., "[Students should know] asking which one moves faster is comparing speed. In a uniform motion, this is actually comparing the slopes of the lines." (Questionnaire Chinese-H-#5)
Meaning of y-value on the graph.(4)	e.g., "[They need to] know what y-value represents on the graph. It is the distance an object moves in given seconds. [They need to know] the y-value at x-value equals zero. This is the starting point, which is very disturbing for kids in this problem." (Questionnaire Chinese-H-#3)
Comparison of numbers and graphs.(3)	e.g., "[Students should] understand what comparisons between graphs mean." (e.g., Questionnaire Chinese-H-#5 &# 8)
Meaning of distance, speed and time and relationship between them.(8)	e.g., "[It is] a classic problem of distance, speed and time. [Students need to] Understand the relationship between these three variables and their representations on graphs." (Questionnaire Chinese-H-#1)
Definition of function.(3)	e.g., "[students need to] understand the definition of function." (e.g., Questionnaire Chinese-H-#10 & #12)
Formula of speed as V=D/T or formula of distance as	e.g., "[They need to know] the formula of speed as V=D/T, or the formula of distance as D=VT to correctly solve this

D=VT.(5)	problem." (Questionnaire Chinese-H-#1)	
Familiar with translation between equations and graphs.(4)	e.g., "[It is] important to understand the translation between equation and graph. They can actually write two functions based on the two graphs. Compare the slopes. The slopes are the coefficients of the linear term of the two equations." (Questionnaire Chinese-H-#7)	
	Student thinking or student misconceptions	
B is higher at t=2 on the graph – B moves faster.(10)	e.g., "[students might be thinking] B is higher at t=2 on the graph which means B moves faster." (e.g., Questionnaire Chinese-H- #1 & #2)	Carelessness.(See Interview Chinese-H-#1, #2, & #4)
B is higher on the graph.(4)	e.g., "[Students would be thinking] B is higher on the graph." (e.g., Questionnaire Chinese – H- #3, #5,& #7)	
B is higher at the starting point.(3)	e.g., Three teachers wrote that "[students might be thinking] B is higher at the starting point." (Questionnaire H-#4, #7, & #9)	
The points on the graph is the speed.(1)	e.g., "[Students might be thinking] speed means those points on the line." (Questionnaire Chinese-H-#6)	
	Correcting approaches	
Provide real-life examples to understand the relationship	e.g., "[I would] ask my students to come up with some examples in order to discuss the relationship between distance, time and speed. [I would] ask a student of	Make connections to students and make math

between distance, time and speed.(4)	my class, 'how far is your home from school?' and 'how many minutes do you usually take to get to school'. [I would] ask two more students the same questions. [I would] record their data and compare these data." (Questionnaire Chinese-H-#7)	learning meaningful.(See Interview Chinese-H-#4)
Discuss the functional relationship between distance, speed and time on the graphs.(3)	e.g., "Discuss the functional relationship between distance, speed and time on the graphs." (Questionnaire Chinese-H-#5)	
Carefully read the graph again to understand the information given in the graph.(6)	e.g., "[I would] ask my kids to carefully read the graph again. [I would ask them to] Identify what is being compared. Students can do it in pairs." (Questionnaire Chinese-H-#1)	Students understand the concept and the content. They need to be very careful when reading graphs.(See Interview Chinese-H-#2 & #4)
Recall the formula of speed and calculate the speed.(1)	e.g., "[I would] ask my students to recall the formula of speed. Use this formula to calculate the speed and then make the comparison." (Questionnaire Chinese-H-	

	#6)

Note. Numbers in parentheses indicate the number of responses that fall into that category.

In total, nine mathematical ideas or concepts were mentioned in teachers' questionnaire responses in this group: the meaning of function graphs in real life; meaning of *x*-value on the graph; slope/rate of change in graph and meaning; meaning of *y*-value on the graph; comparison of numbers and graphs; meaning of distance, speed and time and the relationship between them; definition of function; formula of speed as V=D/T; and translation between equations and graphs.

Eight of the fourteen teachers in this group mentioned in their questionnaire responses that understanding the relationship between distance, speed and time was important. For example, Ms. Zhao wrote on her questionnaire, "[It is] a classic problem of distance, speed and time. [Students need to] Understand the relationship between these three variables and their representations on graphs." (Questionnaire Chinese-H-#1)

Seven teachers pointed out the importance of understanding slope/rate of change in graph and meaning. As Ms. Sun stated in her questionnaire response, "[Students should know] asking which one moves faster is comparing speed. In a uniform motion, this is actually comparing the slopes of the lines." (Questionnaire Chinese-H-#5)

Five teachers in this group mentioned in their questionnaire responses that students need to know the formula of speed. For example, Ms. Zhao wrote, "[They need to know] the formula of speed as V=D/T, or the formula of distance as D=VT to correctly solve this problem." (Questionnaire Chinese-H-#1)

Four of the teachers mentioned the importance of being familiar with translation between equations and graphs in their questionnaire responses. For example, Ms. Li

wrote on the questionnaire, "[It is] important to understand the translation between equation and graph. They can actually write two functions based on the two graphs. Compare the slopes. The slopes are the coefficients of the linear term of the two equations." (Questionnaire Chinese-H-#7)

Additionally, four of the teachers mentioned that students need to pay attention to *y*-value and its meaning, as Mr. Qian wrote on his questionnaire,

> [They need to] know what y-value represents on the graph. It is the distance an object moves in given seconds. [They need to know] the y-value at x-value equals zero. This is the starting point, which is very disturbing for kids in this problem." (Questionnaire Chinese-H-#3)

Three teachers wrote in their questionnaire responses that "[students need to] understand the definition of function," (e.g., Questionnaire Chinese-H-#10 & #12) and three teachers wrote in their questionnaire responses that "[students should] understand what comparisons between graphs mean." (e.g., Questionnaire Chinese-H-#5 & # 8)

Additionally, two teachers mentioned in their questionnaire responses that "[students need to] understand the meaning of function graphs in real life situation," (Questionnaire Chinese-H-#9) and one teacher wrote on her questionnaire that "[students should] know the meaning of v-values on a graph." (Questionnaire Chinese-H-#11)

When asked what they thought their students might be thinking when solving this problem, teachers in this group provided the following responses to the questionnaire. Ten of the teachers in this group wrote that, "[students might be thinking] B is higher at t=2 on the graph which means B moves faster." (e.g., Questionnaire Chinese-H-#1 & #2)

Four of the teachers wrote in their questionnaire responses, "[Students would be thinking] B is higher on the graph." (e.g., Questionnaire Chinese-H-#3; #5; & #7)

Three teachers wrote that "[students might be thinking] B is higher at the starting point," (Questionnaire-H-#4, #7; & #9) and one teacher wrote on the questionnaire that "[students might be thinking] speed means those points on the line." (Questionnaire Chinese-H-#6)

When asked why they thought their students may have this thinking, all four teachers interviewed pointed to students' carelessness. For example, Ms. Zhao explained in her interview, "It is not a difficult problem for my kids. They have seen many examples regarding distance, time and speed. If they make this mistake, it is highly possible that they do not read the graph and the question carefully." (Interview Chines-H-#1)

Similarly, Mr. Qian said in his interview,

> I do not think they do not understand the relationship between distance, time and speed. They are familiar with it. But when it comes to the graph, they need to be more careful. Pay attention to what the x-axis and y-axis are in the graph as well as what is asked in the problem. My kids, I know them. They are sometimes, very careless. Can you say they do not understand? No. They do understand. But this careless habit…I am sometimes really upset… but they are kids. This is what they do at this age. I want them to develop a good learning habit… (Interview Chinese-H-#2)

Ms. Li explained in her interview,

> My kids are smart. I know they can do it…they do not see carefully what is being asked. Seriously, you cannot take it for granted that they understand means they solve problems correctly. A lot times, they know the concept but they do not do the math correctly. (Interview Chinese-H-#4)

When asked how they would help students correct this mistake, the teachers in this group came up with the following approaches according to their responses to the questionnaire.

Six teachers in this group stated in the questionnaire responses that they would ask their students to carefully read the graph and correct this mistake by themselves. For example, Ms. Zhao wrote, "[I would] ask my kids to carefully read the graph again. [I would ask them to] Identify what is being compared. Students can do it in pairs." (Questionnaire Chinese-H-#1)

Ms. Zhao explained in her interview,

> As I said, they have spent quite a long time on the relationship between distance, time and speed. That is an important topic in Physics too…They understand this relationship, they know the equation, they know how to draw graphs to represent the relationship…They are smart. You simply ask them to read the problem and the graph again, and they will understand why they are wrong. I always ask them to be careful when dealing with graphs. Also, I think it is important to help them develop the self-correction strategy when kids make mistake. In my class, I was very surprised at the very beginning a lot of my kids make mistakes on very simple problems as I see. You know they can do very difficult math. Sometimes give solutions that I do not think of. But they fall in some simple traps because

their carelessness. You can change that 'carelessness' through training. I want them to train themselves... (Interview Chinese-H-#1)

Similarly, Mr. Qian wrote in his questionnaire response that, "[I would] ask them to carefully read the graph again and solve the problem by themselves." (Questionnaire Chinese-H-#3)

He explained in his interview,

One way to change their habit of being careless is to ask them to read a problem repeatedly. What did you see from the first time of reading? What did you see from the second time of reading? Did you carefully read it? Did you get enough and correct information to solve the problem. I want them to ask themselves these questions when they are doing math problems… I do not think I have to reteach the content of distance, time and speed. I know they understand it already. They just need to be very careful when reading graphs. (Interview Chinese-H-#2)

Four teachers mentioned that they would provide real-life example to understand the relationship indicated in the graph. For example, Ms. Li wrote on her questionnaire,

[I would] ask my students to come up with some examples in order to discuss the relationship between distance, time and speed. [I would] ask a student of my class, 'how far is your home from school?' and 'how many minutes do you usually take to get to school'. [I would] ask two more students the same questions. [I would] record their data and compare these data. (Questionnaire Chinese-H-#7)

She explained in her interview,

[It is] important to understand what distance means in real life, what time means in real life and what speed means in real life. We are now always emphasizing

making math learning meaningful…I think it make sense for my kids. I want them to come up with their 'real' examples related to this problem. I want them to build that connection… (Interview Chinese-H-#4)

Three teachers mentioned they would "discuss the functional relationship between distance, speed and time on the graphs (*Questionnaire Chinese –H-#5*)." Ms. Sun explained in her interview,

> It is a chance to discuss this type of problems with my kids, I think. I want them to summarize by themselves what distance means, how to represent it on a graph; what time means, how to represent it on a graph; and what speed means, how to represent it on a graph. I want them to discuss what points on a line mean, what starting points mean, etc. I want them to use this mistake as a chance to organize what they learned in my class and summarize it for themselves. It will help their understanding. It will save their time of solving this type of problems. And you know what? When everything is well organized, it is less likely for them to carelessly solve this type of problem… (Interview Chinese-H-#3)

Additionally, one teacher wrote on her questionnaire that, "[I would] ask my students to recall the formula of speed. Use this formula to calculate the speed and then make the comparison." (Questionnaire Chinese-H-#6)

The approaches teachers in this group suggested for correcting this mistake were then categorized into four types. Type 1: Provide real-life examples for students to understand the relationship between distance, time and speed. Type 2: Discuss the functional relationship between distance, speed and time on the graphs. Type 3: Ask

students to carefully read the graph again and correct their mistake by themselves. Type 4: Ask students to recall the formula of speed and calculate the speed.

Curricular knowledge.

In this section, I provide a description of the Chinese higher SES middle school math teachers' curricular knowledge (see Table 4.18). Specifically, I first present these teachers' responses to what instructional materials, textbooks, in particular, they use for teaching the topic of functions. The data source for this is the questionnaires collected from these teachers. I then present the teachers' explanations of how they use these instructional materials, textbooks, in particular, in the classrooms. The data source for this is the interviews. Lastly, I present these teachers' lateral and vertical curriculum knowledge. The data source for both the lateral and vertical curriculum knowledge is the interviews.

Table 21

Chinese Higher SES Teacher Group's Curricular Knowledge

Instructional materials.	Standards (8)
	Renjiaoban Mathematics (13) and *Beijing Kegaiban Mathematics* (1).
	Outside materials (12).
The use of textbooks.	Use some basic examples/problems in the book and complement problems from outside materials.
Functions and other disciplines.	Very few cooperation; some simple physics concepts included.
Functions and other topics in math.	The topic of functions is at the highest level of students' math learning. All the math knowledge is connected.

Note. Numbers in parentheses indicate the number of responses.

According to teachers' questionnaire responses, eight out of the fourteen teachers in this group wrote that they included the *New Standards* in their instructional materials (see e.g., Questionnaire Chinese-H-#1, #2, & #4). Almost all the teachers (thirteen teachers) in this group wrote on their questionnaires that they used the *Renjiaoban Mathematics* which was published by the People's Education Press (see e.g., Questionnaire Chinese-H-#1, #2, & #6). One teacher wrote that she used the *Beijing Kegaiban Mathematics* which was published by the Beijing Publishing Group (Questionnaire Chinese-H-#13). Additionally, almost all the teachers (twelve teachers) in this group wrote that they used outside materials to supple their textbooks in class (see e.g., Questionnaire Chinese-H-#1, #2, & #7).

Based on teachers' interview data, teachers in this group suggested that they used some basic examples or problem in the textbooks and supplemented many higher-level problems from outside materials. For example, Ms. Zhao explained in her interview,

> I think for teachers, the instructional materials are the most important part in teaching, especially in math teaching…Sometimes the problems in the textbook are not up to our expectations, seriously… I am sometimes pissed off by some examples…For some points that should be emphasized in the book, but it does not, so I need to point out for my kids… (Interview Chinese-H-#1)

Another explanation was provided by Ms. Sun. She explained to me in her interview,

> I'd say learning from the textbook is not enough for kids in my school. Let me put it this way. If you understand the problems in the book, you can get probably sixty-percent of the points on the test, especially on the high school entrance exam.

But for my kids, their goal is not this sixty-percent. Instead, their goal is one-hundred percent. The problems in the book are very easy for them to solve, seriously. They need more practice on the higher-level problems, so I collect those problems for them from my own resources as well as from some other teachers' resources. I use these problems in my class. (Interview Chinese-H-#3)

Similar explanation can also be found from Ms. Li's interview. She explained to me,

I do use some examples from the textbook. Those examples are designed, I think, following the requirements in the New Standards. So, understanding the content in the textbook is basic for us to achieve. But we have school district exams to pass and more importantly we have the high school entrance exam at the end of middle school education. 'Basic' level cannot send them to 'good' high schools…en, our kids are expected to go to all those 'good' high schools. This means they need to succeed on the entrance exam. That pressure is on both teachers and kids.... Yeah, we complement problems at the higher demand level. These are not in the textbook… (Interview Chinese-H-#4)

According to teachers' interviews in this group, when asked about the interdisciplinary work they might have done, teachers explained that they had done a very limited number of interdisciplinary collaborations with other teachers. For example, Mr. Qian said,

…I understand it might be better to collaborate with teachers from other disciplines for some particular topics in math, but in reality we rarely do it, honestly. Sometimes I do incorporate some physics concepts into my teaching.

For example, distance and speed problems. Volume problems. En, pressure problems. But not that much… (Interview Chinese-H- #2)

In another example, Ms. Sun said in her interview,

…I mean math is kind of the foundation… For example, you need to know the basic calculations for physics and chemistry, right? But in my teaching, we do not really have to incorporate other disciplines except for the contexts of the problems. For instance, the context of the problem is the about the pressure. This can be related to physics… That's all we do. (Interview Chinese-H-#3)

In the interviews, when asked to describe the connections between functions and related topics in math learning, teachers suggested that the topic of functions is at the highest level of math learning and all the math knowledge is connected. For example, Ms. Zhao said,

I think, the topic of functions is actually at the highest level of secondary math learning. I mean not only for middle school, but also for high school math learning. In my opinion, math learning can be divided into four levels. The first level is about learning numbers; the second level is about learning expressions or equations and inequalities; the third level is about learning sets of equations and inequalities; and the fourth level is about learning change and correspondence, variables and functions. From this perspective, functions are at the highest level of math learning…so the learning of functions is built upon all the three levels of math learning, from numbers to equations, set of equations, etc. all these topics are connected. Math is an entire system with all the knowledge points connected to each other… (Interview Chinese-H-#1)

Similarly, Ms. Sun said,

> …Functions are the most important topic in secondary school math learning…How are these topics connected? How is students' math knowledge growing? It is from the learning of numbers. From numbers to expressions. Then we have equations, inequalities… when we try to investigate how things change, we bring in functions…and then we want to visualize functions, we bring in the coordinate plane and function graphs. Functions then are connected to geometry…functions can relate to almost everything we teach in the secondary math learning. It is important for middle school, and even more important for high school math learning. Actually math is a connected network. All those topics or issues are located at different points in this network, but they connect to each other from different paths… (Interview Chinese-H-#3)

The Chinese Lower SES Middle School Math Teacher Group

Teachers' instructional knowledge for introducing the topic of functions.

In this section, I provide three sub-sections to describe the Chinese lower SES middle school math teachers' instructional knowledge for introducing the topic of functions. These subsections are the goal of teaching introductory class of functions, the construction of mathematical tasks and the use of representations of functions. In each subsection, a description of teachers' instructional decisions is presented based on the data collected from lesson plans, followed by teachers' explanations for their instructional decisions which are based on the data collected from semi-structured interviews. Fourteen introductory lesson plans were collected from the Chinese lower

SES middle school math teacher group and five of the teachers in this group were interviewed. All the quotes in this section were translated by the researcher.

The goals of teaching introductory class of functions.

The Chinese lower SES middle school math teachers' goals of teaching introductory class of functions covered nine aspects. They expected their students to be able to understand the relationship that function is one input having exactly one output; change; the concept of constant and variable; linear functions; independent and dependent variable; algebraic representation of functions; coordinate plane; graphical representation of functions; and multiple representations of functions. See Table 22.

Table 22

The Goals of the Introductory Class for the Chinese Lower SES Teacher Group

Goal	Number of response	Examples	Teachers' explanations
Function definition as one input with exactly one output.	12	e.g., "In this lesson, students will understand the definition of function. Function is a rule that assigns to each independent variable exactly one dependent variable." (Lesson plan Chinese-L-#7)	Standards and textbooks.
Change.	2	e.g., "Students will understand what change means in math." (Lesson plan Chinese-L-#14)	
Constant and variable.	11	e.g., "Students will understand what constant and variable means. They will use these two concepts to understand what a function is."	Sequence the curriculum and textbook (and student

		(Lesson plan Chinese-L-#5)	ability).
Linear functions.	3	e.g., "Understand what a linear function is." (Lesson plan Chinese-L-#7)	
Independent and dependent variable.	6	e.g., "Students will understand the concept of independent and dependent variable. They will be able to identify the independent variable and the dependent variable given a function." (Lesson plan Chinese-L-#3)	Help understand a function; standards and textbook (and student ability).
Algebraic representation.	5	e.g., "Students will be able to understand algebraic functions. They will be able to write the algebraic form of a function in a problem." (Lesson plan Chinese-L-#11)	The most important representation; difficult for kids on tests.
Coordinate plane.	1	e.g., "Students will be able to understand what a coordinate plane means. They will be able to identify x-axis and y-axis of a coordinate plane for a problem." (Lesson plan Chinese-L-#12)	
Graphical representations.	3	e.g., "Students will understand the graphical representation of functions. They will understand how function graphs represent functions." (Lesson plan Chinese-L-#5)	Students are going to have difficulties in graphs.
Three forms of	5	e.g., "They will understand three forms of representations of	Standards and

representation.	functions. These are algebraic form, graphical form and tabular form." (Lesson plan Chinese-L-#8)	textbook.

In the fourteen introductory lesson plans collected from this group, twelve of them mentioned in their lesson plans that they expected their students to understand the definition of function as one input with exactly one output. For example, a teacher wrote, "In this lesson, students will understand the definition of function. Function is a rule that assigns to each independent variable exactly one dependent variable." (Lesson plan Chinese-L-#7)

For another example, Ms. Zhou wrote in her lesson plan, "They will understand the definition of function and use the definition to identify functions in reality." (Lesson plan Chinese-L-#3)

When asked why they included the definition of function as important goal for this introductory lesson, teachers who were interviewed in this group pointed to the *New Standards* and textbooks. For example, Ms. Zhou explained in her interview,

> This is the curriculum. If I go to the eighth grade curriculum map and if I go the chapter of functions, it is explicitly state there that students are expected to understand the definition of function as a rule that assigns each input with exactly one output. And if I go to our textbook, the definition of function is also stated… (Interview Chinese-L-#1)

Similarly, Mr. Wang explained in his interview, "Understanding the definition of function is required by our standards and textbooks… I mean we are asked to follow what's asked by the standards and textbooks. That's the guide." (Interview Chinese-L-#4)

Eleven teachers in this group mentioned in their lesson plans that they expected their students to understand the concept of constant and variable. For example, Mr. Wu wrote, "Students will understand what constant and variable means. They will use these two concepts to understand what a function is." (Lesson plan Chinese-L-#5)

Mr. Wang wrote in his lesson plan, "Students will be able to understand the concept of constant and variable and to identify constant and variable in real-life situations." (Lesson plan Chinese-L-#4)

When asked why they included the concept of constant and variable as important goal for this introductory lesson, teachers, again, pointed to the *New Standards* and textbooks. For example, Mr. Wu said,

> This is our book…uh, if you look at the chapter of function, you will see constant and variable in the first section. It is not a very difficult section, but sometimes I have to spend two lessons on just constant and variable until they get it…it is important for us to follow the curriculum, follow the book. These kids are low. I know some other schools… they are much faster. They can even skip this section, but we cannot." (Interview Chinese –L-#2)

Similarly, Mr. Wang said, "This is a goal stated in the *Standards*. If you are going to introduce the concept of function, you will have to be prepared for introducing the concept of constant and variable. This is written in our textbook, too." (Interview Chinese-L-#4)

Six of the fourteen teachers mentioned that understanding independent and dependent variable was an important goal in their introductory classes. For example, Ms. Zhou included this goal in her lesson plan, "Students will understand the concept of

independent and dependent variable. They will be able to identify the independent variable and the dependent variable given a function." (Lesson plan Chinese-L-#3)

When interviewed why she included this goal, Ms. Zhou explained,

> I think distinguishing between independent and dependent variable is important for understanding a function in the first class. That's the first step in learning function. If you can tell which one is independent variable and which one is dependent variable, you will be able to tell the relationship represented in a function. A is changing because B is changing. Not vice versa. I need them to understand this at the very beginning. Otherwise they will find it hard to figure out the relationship in a function. (Interview Chinese-L-#1)

Mr. Wang also included understanding independent and dependent variable as a goal in his lesson plan. He wrote, "Students are expected to understand the meaning of independent and dependent variable, be able to identify independent and dependent variable in problems and be able to understand a function from the perspective of independent and dependent variable." (Lesson plan Chinese-L-#4)

He explained his goal in the interview,

> …It is explicitly stated in the standards. Here. Students are expected to understand what independent variable means and what dependent variable means and to be able to identify them in a relationship. In my experience, my kids had a bit problem on distinguishing them at first in the past few years, but they got it after a couple of lessons…I would like to address this at the very beginning." (Interview Chinese-H-#4)

Understanding representation of functions was explicitly stated as important goal by a lot of teachers in this group. Five teachers mentioned in their lesson plans that it was important for students to be able to understand multiple representations of functions in their introductory classes. For example, Ms. Feng wrote in her lesson plan, "They will understand three forms of representations of functions. These are algebraic form, graphical form and tabular form." (Lesson plan Chinese-L-#8)

When asked to explain this goal in the interview, Ms. Feng said, "It is stated in our standards and textbook that students are expected to be able to use three forms of representation of function. They are equations, tables and graphs." (Interview Chinese-L-#5)

In another lesson plan, Ms. Zheng also stated this goal, "Students will understand the representations of functions. They are expected to use three most frequently used forms – algebraic functions, tables and graphs." (Lesson plan Chinese-L-#10)

She explained this goal to me in the interview,

> It is an important goal written in this book that students should be able to understand functions from three forms of representation. They are expected to be able to write simple equations, create tables of values and draw simple graphs such as lines for function in the introductory classes. (Interview Chinese-L-#3)

Five teachers explicitly stated in their lesson plans that students were expected to be able to write equations for simple functions and get y-values for particular x-values using equations in the introductory class. For example, one teacher wrote, "Students will be able to understand algebraic functions. They will be able to write the algebraic form of a function in a problem." (Lesson plan Chinese-L-#11)

Ms. Zhou also included a similar goal, "Students are expected to write the algebraic form for a function." (Lesson plan Chinese-L-#3)

When asked why she included this in her lesson plan, Ms. Zhou explained in the interview,

> Equations are important in learning functions…yes, not every function can come up with an equation, but still, equation is the most important form of representation of functions. You cannot deny this… It is difficult for my kids, uh, especially when it comes to those question items related to algebraic functions on the high school entrance exam. (Interview Chinese-L-#1)

Ms. Feng also included the goal, "Students will understand the algebraic form of a function." (Lesson plan Chinese-L-#5) She explained in the interview,

> Algebraic function is the most important form of representation of function. I told my kids that they can understand a function through its algebraic form at the first class. I think it is important for them to raise their awareness of algebraic functions. Almost all the question items on functions appearing on the high school entrance exam are, more or less, related to algebraic functions. Write the algebraic function, get x- or y-values using algebraic function, transfer between algebraic function and graphs. All refer to algebraic functions. (Interview Chinese-L-#5)

Three teachers explicitly stated in their introductory lesson plans that students were expected to be able to understand graphical representations of functions. For example, Mr. Wu wrote in his lesson plan, "Students will understand the graphical

representation of functions. They will understand how function graphs represent functions." (Lesson plan Chinese-L-#5)

> When asked why he included this goal, Mr. Wu explained in the interview, Function graphs seem easy to understand at the start. Yes, kids have fun in drawing. As we move along, however, the difficulty level of understanding function graphs increases sharply, seriously. My kids always find it extremely difficult for them to read a graph, to get information, and to use it to solve a math problem, especially when it comes to parabolas, equations and graphs, etc. I want to address the importance of graphical representation. I want to build their confidence in dealing with graphs from the first class. Do not be afraid of graphs. Do not be afraid of problems involving graphs and equations. (Interview Chinese-L-#2)

Three teachers mentioned in their lesson plans that students were expected to "understand what a linear function is," (Lesson plan Chinese-L-#7) or "understand what a linear function means from the equation and the graph." (Lesson plan Chinese-L-#13)

Two teachers mentioned in their lesson plans that students should understand the concept of change. For example, one of them wrote, "Students will understand what change means in math." (Lesson plan Chinese-L-#14)

One teacher mentioned in her lesson plan that her students were expected to understand the coordinate plane. She wrote, "Students will be able to understand what a coordinate plane means. They will be able to identify x-axis and y-axis of a coordinate plane for a problem." (Lesson plan Chinese-L-#12)

The construction of mathematical tasks.

Table 23

Mathematical Task by Cognitive Demand Level for the Chinese Lower SES Teacher Group

Cognitive demand of mathematical task	Example	Teachers' explanations
	Lower-level demand	
Memorization.(4)	e.g., In a changing process, we call the quantities not changing _____; we call the quantities changing _____. (Lesson plan Chinese-L-#4)	Student's lower ability level; students cannot really understand concepts at this age; lack of prior knowledge.
Procedures without connection to concepts/understanding/meaning.(0)	N/A	
	Student thinking or student misconceptions	
Procedures with connection to concepts/understanding/meaning.(25)	e.g., You are asked to use a 10cm long rope to make a rectangle. How does the area of the rectangle change when you change the length of the width of the rectangle? Can you make a table to record the area of the rectangle at different lengths of the width? If we set the width as x, the area as S, could you write a function	Consider the new standards challenge and high school entrance exam.

	for S? (Lesson plan Chinese-L-#5)
Doing mathematics.(0)	N/A

Note. Numbers in parentheses indicate the number of responses that fall into that level.

In total, twenty-nine mathematical tasks were constructed in the introductory lesson plans in this group. Four out of the twenty-nine mathematical tasks were constructed at the level of memorization. An example of these mathematical tasks is:

Write the equations for the following relationships:

The length of a side and the area of a square;

The perimeter of a circle and the circumference of a circle; and

The Pythagorean Theorem. (Lesson plan Chinese-L-#5)

In a changing process, we call the quantities not changing _____; we call the quantities changing _____. (Lesson plan Chinese-L-#4)

In the interview, teachers explained their reasons of constructing this type of mathematical tasks for their students as follows.

Ms. Zhou said in her interview,

These kids are low…I don't think they really understand the concept of function, especially at the introductory class. They may know how to write equations or play with graphs better as we move along…I just doubt if they really understand a concept…what a function really means to them. Sometimes they need to memorize the definition (Interview Chinese-L-#1)

Mr. Wu said in his interview,

I don’t think they can really understand given the ability level they are and the age they are. You know? Concepts are actually something most difficult to grasp for

kids. Even we teachers sometimes do not know how to explain a concept…I want them at least to memorize what a function is, how to express a function using an equation, or how to draw a function graph. For these kids, you do need to ask them to memorize and to practice. I expect them to know rather than understand something at the beginning. They will forget a lot of stuff as they move to new stuff anyway… I do want to include some memorize type of questions in my class…It works. (Interview Chinese-L-#2)

Mr. Wang said in his interview,

I need to have some questions aimed to make up for their prior knowledge since these kids are not as good compared to kids in some other top schools. We do have some students, you know, uh…who still do not know how to get the area of a circle, which they should've learned in the elementary school…functions is not an isolated topic. It can actually connect to a lot of stuff they have already learned. They simply forget or they did not get it at the time they learned. They are lack of the foundation. And you know they are still kids. They have not built abstract thinking, compared to high school kids. It is hard for them to understand such a concept... I mean sometimes I do need to make them memorize some stuff. (Interview Chinese-L-#3)

The other twenty-five mathematical tasks were constructed at the level of cognitive demand focused on procedures, with connections to concepts, understanding or meaning. Some examples of this type of mathematical tasks are as follows,

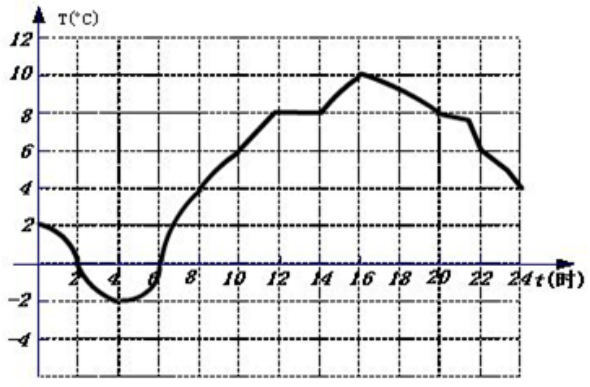
A car is moving at the speed of 60 kilometers per hour. The distance is expressed as s, the driving time is expressed as t. Fill in the following table,

t (h)	1	2	3	4	5
s (kilometer)					

Can you write an equation for the relationship between the distance and the time the car moves? What do you think is changing? What do you think is not changing? Can you draw a graph based on the given information? Explain the graph to your shoulder partner. (Lesson plan Chinese-L-#3)

You are asked to use a 10cm long rope to make a rectangle. How does the area of the rectangle change when you change the length of the width of the rectangle? Can you make a table to record the area of the rectangle at different lengths of the width? If we set the width as *X*, the area as *S*, could you write a function for *S*? (Lesson plan Chinese-L-#5)

This is a graph indicating the change of temperature during 24 hours in Beijing.



What is the temperature at the 8th hour? What about the 14th hour and 22nd hour? What is the highest temperature during this day and what is the lowest?

How does the temperature change between the 4th and 12th hour? How does the temperature change between the 12th and 14th hour? How does it change between the 16th and 24th hour?

How do you think the temperature changes based on your observation?

(Lesson plan Chinese-L-#7)

All the five teachers who were interviewed in this group mentioned the *New Standards* and the high school entrance as the major reason of construction this type of mathematical task. For example, Ms. Zheng said in her interview,

> I've basically followed those requirements in the new standards. The new standards ask for the understanding and meaning. Simple calculations or procedures are rarely seen on the high school entrance examination, either. You will not see a question simply asking you to get a *y*-value given a particular *x*-value. You have to understand the problem first. Find out the relationship, sometimes it can be expressed as an equation, sometimes as a graph. You have to make predictions based on your understanding… (Interview Chinese-L-#3)

Similar response can be found in Ms. Feng's and Ms. Zheng's responses. Ms. Feng said in his interview,

> We have two guides: the new standards and the high school entrance exam. These are our two flags… they test students' ability. What is ability? I think it is about when they are given a problem which they may or may not see before, they can read… can understand what this problem asks for. They can connect to the math knowledge they have learned to this problem. Use math language to solve it. Apply what they learn to a new situation. This relies on understanding. Not

knowing, but real understanding. How do they develop this ability? We teach them math… help them develop math skills. We train them. See the task I include in this class? This is one way to train them. Get familiar with this type of problems… (Interview Chinese-L-#5)

Ms. Zheng explained in her interview,

I cannot just ask them to memorize the rules or do all the calculations. It will not help them on the test. The complexity of calculation was only emphasized in the old standards and test. Now not anymore… they need to know why, how… Why we call this function, this independent variable, that dependent variable? How we make meaning of a function equation? Can you make up a real-life story? Can you use a graph to illustrate a function…What the most important part is how you deal with information, which is highly addressed in the high school entrance exam. You need to know how to take the information from a problem. What does it mean? How you use the information to infer, how you relate to the knowledge points you've learned. Otherwise you are gonna fail on the exam… (Interview Chinese-L-#4)

Teachers' use of representations of functions.

Teachers in this group used three forms of representations of functions in total in their introductory lesson plans. These forms are equations or algebraic functions, tables and graphs. Some examples of representations used in their lesson plans are shown as follows.

Equations:

For example,

Understand the algebraic form of functions such as s=60t, y=10x, and L=10+.5m.

(Lesson plan Chinese-L-#1)

Some of the equations below are functions. Do you know which ones are functions and which ones are not? Explain your answer.

(1) $xy = 2$; (2) $x + y = 5$; (3) $|y| = 3$;

(4) $x^2 + y^2 = 10$; (5) $y = x^2 - 4x + 5$

(Lesson plan Chinese-L-#6)

Tables:

For example,

Radius	2	4	5	7	8
Area					

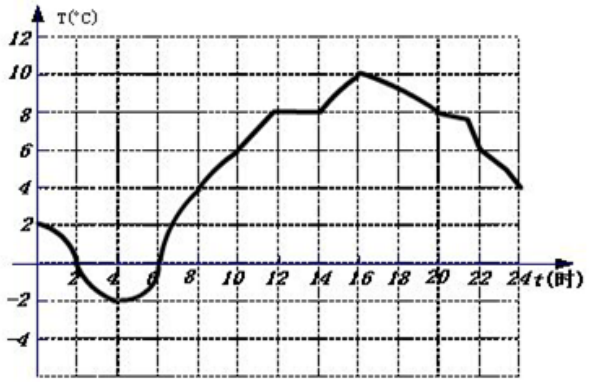
(Lesson plan Chinese-L-#5)

Graphs:

For example,

This is a graph indicating the change of temperature during 24 hours in Beijing.

(Lesson plan Chinese-L-#7)



How did teachers in this group use these forms of representations of functions in their lesson plans? Among the fourteen teachers whose lesson plans were collected in this group, all of them used equations in their lesson plans (see Table 24). Three of the teachers used only equations in their lesson plans. Three of them used both equations and graphs in their lesson plans. Five teachers used both equations and tables in their lesson plans. Three teachers used equations, tables and graphs in their lesson plans.

Table 24

The Use of Representations of Functions by the Chinese Lower SES Teacher Group

Form of representation	Number of response
Equation only.	3
Equation and graph.	3
Equation and table.	5
Equation, table and graph.	3
Teachers' explanations:	
Equation, table and graph are three forms of representations.	
Equation is the most important form of representations of functions.	
Tabular and graphical representations are important in understanding functions.	

In their interviews, when asked to explain their attitudes toward the use of representations of functions in the introductory classes, teachers provided some explanations as follows.

They stated in their interviews that equations, tables and graphs are three forms of representations which these teachers expected their students to understand. For example, Ms. Zhou said in her interview, "I require my students to be able to use equation, table and graph of functions. Understanding these three forms of representations are explicitly stated in the new standards and in our textbook." (Interview Chinese-L-#1)

Ms. Zheng had a similar explanation when she was interviewed,

> When it comes to forms of representations of functions, equation, graphic and tabular representations are the three representations that our standards and textbooks require our students to understand. I expect them to be able to write an equation, create a table of values and draw a graph for a function… (Interview Chinese-L-#3)

These teachers also gave me some explanations showing that they expected their students to understand that equations were the most important form of representations of functions. For example, Ms. Zhou explained in her interview,

> Equations are the most important representation that I require my students to understand…Writing functional equations is tested most frequently on the high school entrance exam, which is usually difficult for my kids…If they can understand an equation of a function, other forms like graph and table will not be a problem for them, I think. (Interview Chinese-L-#1)

Similarly, Ms. Feng explained in her interview,

> I think equation is the most important and difficult one…Once they understand the equation of a function, they know all the features of a particular function. It is the key. If you see our test sheets, you will find that usually the most difficult one

is writing an equation from a graph with some information given in a scenario… (Interview Chinese-L-#5)

Additionally, they provided some explanations indicating tables and graphs were important in helping students understand functions. For example, Mr. Wu explained to me in his interview,

> We use also tables and graphs, especially graphs, to help students understand a function, to understand what a function looks like on a coordinate plane. Tables and graphs are visual. Kids like drawing tables and graphs. In the introductory class, we use tables a lot. That is an easy way to understand a function and to write an equation. As we move along, we do not use tables that much. We use graphs a lot. We create many question items testing students' ability of using equations and graphs of functions together. (Interview Chinese-L-#2)

Ms. Zheng stated in her interview,

> Tables and graphs are two important forms we ask our students to grasp. Equations are very abstract for kids, but tables and graphs are very visual. For example, you can really fill in numbers in a table of values and kids like real numbers. For another example, you can see what a function really looks like if you have a graph – a line, a parabola, any curve…so we use tables and graphs to help our students understand functions. They are also…like important supplement of equations. (Interview Chinese-L-#3)

Teachers' knowledge of student understanding of functions.

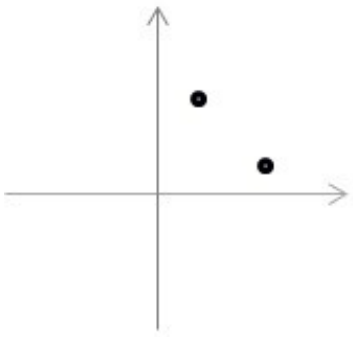
In this section, I provide detailed description of teachers' responses to two scenarios of students' mistakes. The first scenario is about students' mistakes on drawing

function graphs through two given points on a coordinate plane. The second scenario is about students' mistakes in comparing two linear functional graphs on a coordinate plane. For each scenario, I describe teachers' knowledge of students' understanding of functions from three aspects, based on their responses in the questionnaire. First of all, I describe the mathematical ideas that teachers think are important for students to correctly solve the math problem; second, I describe the "thinking" these teachers suggest their students might have leading to the mistake; and third, I describe teachers' approaches to correcting their students' mistakes. In each of the aspects, a description of teachers' responses is presented based on the data collected from questionnaires, followed by teachers' explanations collected from their semi-structured interviews. In total, fourteen questionnaires were collected from the Chinese lower SES middle school math teacher group and five of the teachers in this group were interviewed.

Scenario 1.

A review of the scenario:

If a student is asked to give an example of a graph of a function that passes through the points A and B, she gives an example as shown in Figure 2. When asked if there is another answer for this question, she says "No".



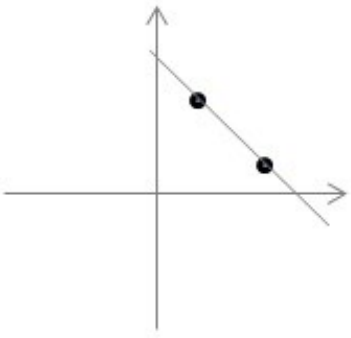


Table 25

Chinese Lower SES Teachers Dealing with Students' Mistake on drawing Functional Graph(s)

Teachers' response	Example	Teachers' explanations
	Mathematical ideas	
Function definition: one input exactly with one output.(9)	e.g., "[This problem] it is actually testing students' understanding of function definition. Students need to understand the definition to avoid falling in the 'trap'." (Questionnaire Chinese-L-#6)	
Function graphs come in many shapes.(6)	e.g., "[Students need to] understand that functions have many shapes, such as lines, parabolas, curve and many others." (Questionnaire Chinese-L# 9)	
Non-linear graphs.(5)	e.g., "Students were exposed to non-linear graphs at the introductory lessons. [They need to] remember those linear graphs to correctly solve	

	the problem here." (Questionnaire Chinese-L-#9)	
Linear function features.(1)	e.g., "[students need to] know linear-function features in order to correctly solve the problem." (Questionnaire Chinese-L-#13)	
Equation determines the shape of a graph.(3)	e.g., "[Students need to] understand that equation is the most important representation for functions. For middle school students, quadratic function is the non-linear function on which students spend a large amount of time. If they can write a quadratic equation, they know it also can be a parabola through two points on a coordinate plane. They should be very familiar with equations." (Questionnaire Chinese-L-#2)	
	Student thinking or student misconceptions	
Connecting two points create a line.(8)	e.g., "[Students might be thinking] two points make a line on a coordinate plane." (e.g., Questionnaire Chinese-L-#1, #5, & #7)	Fixed-thinking after learning linear functions. (See Interview Chinese-H-#1) Not enough experience with non-linear functions. (See Interview Chinese-H-#3 & #4)
A function is a line (a linear function).(3)	e.g., "[Students might be] only thinking lines when they saw two points given in the graph."(Questionnaire Chinese-L-#2)	
Do not understand	e.g., "May not understand the	

the problem.(3)	question that this problem asked." (Questionnaire Chinese-L-#4)	
	Correcting approaches	
Directly use non-linear graphs.(6)	e.g., "[I would] choose to directly use nonlinear functions as counter-examples to correct students' misconception of 'connecting two points create a line'." (Questionnaire Chinese-L-#1)	Visual and straight for kids- s an appropriate approach for lower kids to understand the problem.(See Interview Chinese-L-#1)
Reteach the definition of function and its three representations.(5)	e.g., "[I would like] to teach my students the definition of function again. [I would] also teach them the three forms of representations of functions again to correct their mistake." (Questionnaire Chinese-L-#6)	Kids may not be able to understand the concept of function at the introductory classes. (See Interview Chinese-L-#4)
Make up stories.(1)	e.g., "Make up some stories of linear and nonlinear functions and ask students to tell the difference." (Questionnaire Chinese-L-#11)	
Use equation of non-linear functions the then graph them.(2)	e.g., "Use nonlinear equations and graph them on the coordinate plane." (Questionnaire Chinese-L-#2)	Once students understand equations, they can easily understand functions graphs. (See Interview Chinese-L-#2)

Note. Numbers in parentheses indicate the number of responses that fall into that category.

Five mathematical ideas (or concepts) were mentioned as important for students to know to correctly solve the problem in teachers' questionnaire responses: function definition; non-linear functions and graphs; functions come in many shapes; linear-function features; and equation determines the shape of a graph.

Nine out of the fourteen teachers in this group mentioned that students need to know the definition of function to correctly answer the question in the scenario. As Mr. Wang wrote in his questionnaire response, "[This problem] it is actually testing students' understanding of function definition. Students need to understand the definition to avoid falling in the 'trap'." (Questionnaire Chinese-L-#6)

Six teachers mentioned the importance of understanding functions coming in many shapes. For example, Ms. Feng wrote in her questionnaire response, "[Students need to] understand that functions have many shapes, such as lines, parabolas, curve and many others." (Questionnaire Chinese-L# 9)

Five teachers mentioned the importance of knowing non-linear graphs, as Ms. Feng wrote in her questionnaire response, "Students were exposed to non-linear graphs at the introductory lessons. [They need to] remember those linear graphs to correctly solve the problem here." (Questionnaire Chinese-L-#9)

Three teachers wrote on the questionnaire that understanding equation determines the shape of a graph. As Mr. Wu wrote in his questionnaire response,

> [Students need to] understand that equation is the most important representation for functions. For middle school students, quadratic function is the non-linear function on which students spend a large amount of time. If they can write a

quadratic equation, they know it also can be a parabola through two points on a coordinate plane. They should be very familiar with equations. (Questionnaire Chinese-L-#2)

One teacher mentioned in his questionnaire response that "[students need to] know linear-function features in order to correctly solve the problem." (Questionnaire Chinese-L-#13)

When asked what thinking would potentially lead students to make this mistake, teachers provided the following responses to the questionnaire.

Eight of them wrote in their questionnaire responses that "[students might be thinking] two points make a line on a coordinate plane." (e.g., Questionnaire Chinese-L-#1, #5, & #7)

Three of the fourteen teachers mentioned in their questionnaire responses that "[students might be] only thinking lines when they saw two points given in the graph." (Questionnaire Chinese-L-#2)

Three teachers stated in their questionnaire responses that their students "may not understand the question that this problem asked." (Questionnaire Chinese-L-#4)

When asked why they thought their students might have this thinking, teachers gave the following explanations according to their interviews. For example, Ms. Zhou explained in her interview,

> You know, our kids are lower compared to a lot schools. They have that…we call fixed thinking. We teach them 'connecting two points create a line', they memorize it. They use it everywhere afterwards. I mean, they do not really think. Like this problem. They see two points, Okay, my teacher told me 'two points

make a line', then draw a line. This is probably their thinking process. It is what we called fixed-thinking problem… (Interview Chinese-L-#1)

Mr. Wang explained in his interview,

I think it is probably because they do not have enough exposure to non-linear functions. For eighth graders, all they familiar with are linear functions…uh, they did learn proportional and inverse-proportional functions, but I do not think they practice a lot on inverse-proportional…they only think of linear when they see this problem I guess. (Interview Chinese-L-#4)

Ms. Zheng, who mentioned that her students might be thinking functions as only linear functions or lines, explained in her interview,

I think one reason might be the lack of exposure to non-linear functions. For my eighth graders, they do a lot on linear functions. Another reason might be the fixed thinking. I mean for ninth graders, they learn quadratic and practice a lot on it. If they still make this mistake, it indicates a fixed-thinking process. I mean, we did teach them 'connecting two points create a line on a coordinate plane' when teach the graph of linear function. But that only works for linear functions. When we were under the topic of linear, we did not mention any condition to apply this 'rule'. But the problem asked in the scenario does not say it is a linear, but kids still in that thinking process…they take it for granted. (Interview Chinese-L-#3)

Mr. Wu, who thought his students may not really understand what the problem asks for, explained to me in his interview, "I doubt if they really understand the problem. This problem is a bit like a trap…you know? Trap-type of problem… I think they do not

know what the problem really asks for. It asks them to connect the two points, uh, not asks them to draw a line…" (Interview Chinese-L-#2)

When asked how they would help their students correct this mistake in the scenario, teachers in this group provided several approaches according to their questionnaire responses. Six teachers mentioned that they would directly provide students with nonlinear graphs. For example, Ms. Zhou wrote in her questionnaire response, "[I would] choose to directly use nonlinear functions as counter-examples to correct students' misconception of 'connecting two points create a line'." (Questionnaire Chinese-L-#1)

> When asked why she selected this approach, Ms. Zhou explained in the interview,
>
> They [students] made this mistake basically because they did not see many non-linear function graphs. You give them examples of non-linear graphs and they will know there is not 'one line' between two points. Very visual and straight for my kids, I think, and this is an appropriate approach for them to understand the problem. (Interview Chinese-L-#1)

Ms. Zheng wrote in her questionnaire response, "[I would] draw a parabola going through these two points. They will see, 'Okay, it is a function, it goes through these points', and they will know it is not a line." (Questionnaire Chinese-L-#5)

She explained in her interview,

> I think it is an easy way to make my students realize where their mistake is. Yes, they did not consider non-linear functions. Once you give them those counter-examples, they will know they can draw as many as they can to connect two points. I am not sure this type of question will be asked on the high school entrance exam, but I do want them to know they can only apply the rule

'connecting two points create a line' under the condition of linear functions. (Interview Chinese-L-#3)

Five teachers mentioned the definition of function in correcting the mistake in their questionnaire responses. For example, Mr. Wang wrote on his questionnaire, "[I would like] to teach my students the definition of function again. [I would] also teach them the three forms of representations of functions again to correct their mistake." (Questionnaire Chinese-L-#6)

He explained his approach in the interview,

> Once they understand what a function is, they will not have this misconception…I use this approach based on my students' ability level. I mean I think my kids may not be able to understand the concept of function at the introductory classes. As we move onto linear functions. They deal a lot with the procedures…they forget the definition of function. Functions can be anything. Functions can be represented in equations, graphs, tables, words…Surely we emphasize the first three forms, especially equation representations. So I think it is a good chance to pick up the concept using this 'mistake'... (Interview Chinese-L-#4)

Two teachers wrote on the questionnaire that they would "use nonlinear equations and graph them on the coordinate plane." (Questionnaire Chinese-L-#2) Mr. Wu explained this approach in his interview that,

> I would use equations and graph them. They know inverse-proportional functions and quadratic functions. They should be very familiar with quadratic functions since the high school entrance exam tests a lot on it. I want them to understand equations first. Once they get equations, it will not be that hard for understanding

functions. In my opinion, equations stand at a higher level of thinking… The two points can be found on a line and also be found on a parabola. If you have a quadratic equation, you will have a corresponding graph… (Interview Chinese-L-#2)

Additionally, one teacher wrote in her questionnaire response that she would "make up some stories of linear and nonlinear functions and ask students to tell the difference." (Questionnaire Chinese-L-#11)

Teachers' approaches to correcting students' mistake in this group were then categorized into four types. Type 1: Directly use nonlinear function graphs. Type 2: Reteach and discuss the definition of function. Type 3: Make up stories for linear and nonlinear functions and graph them. Type 4: Provide examples of nonlinear function equations and graph them.

Scenario 2.

A student is given the position vs. time graph as presented below. When asked to compare the speeds of the objects at time $t = 2$ sec., the student responds by saying that Object B is moving faster.

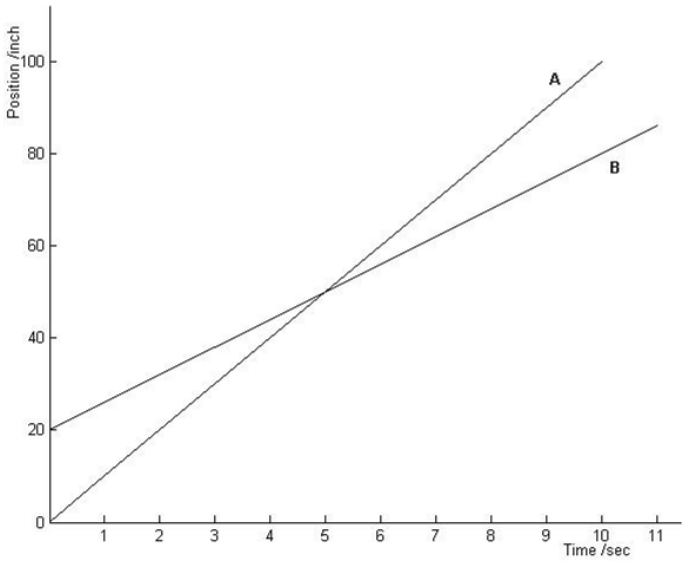


Table 26

Chinese Lower SES Teachers Dealing with Students' Mistake on Comparing Linear Functions on A Coordinate Plane

Teachers' response	Example	Teachers' explanations
	Mathematical ideas	
Understand the meaning of function graphs in real life.(7)	e.g., "[Students need to] understand what a graph means in real life. A graph can represent a relationship between time point and temperature, a graph representing a relationship between the diameter and the area of a circle. Students need to make these connections. For this problem, it is about motion." (Questionnaire Chinese-L-#1)	
Meaning of x-value and y-value on the graph.(2)	e.g., "[students need to] understand the meaning of x-values on the graphs and the meaning of y-values on the graph."	

	(Questionnaire Chinese-L-#10)	
Slope/rate of change in graph and meaning.(3)	e.g., "Slope is a very important concept to solve this problem. It is a comparison of slopes. Students need to understand what a slope means and how to determine it." (Questionnaire Chinese-L-#2)	
Meaning of distance, speed and time and relationship between them.(9)	e.g., "The key of this problem is an understanding of the relationship between distance, speed and time and their representations on the graph." (Questionnaire Chinese-L-#5)	
Definition of function.(2)	e.g., "Understand the definition of function." (Questionnaire Chinese-L-#13)	
Equation of speed as V=D/T.(4)	e.g., "[Students need to] know how to get speed from a problem. They are supposed to remember the formula V=D/T." (Questionnaire Chinese-L-#6)	
Familiar with translation between equations and graphs.(1)	e.g., "[Students need to] be able to translate between equations and graphs of functions." (Questionnaire Chinese-L-#8)	
	Student thinking or student misconceptions	
B is higher on the graph. (9)	e.g., "[students might be thinking] B is higher on the graph." (Questionnaire Chinese-L-#1 & #2)	Do not know how to read a graph on a coordinate plane (See Interview Chinese –L-#1,
B is higher at t=2 on the graph – B moves faster. (2)	e.g., "[Students might be thinking] B is higher at t=2 on the graph which means B moves faster." (e.g., Questionnaire	

	Chinese-L- #8; #9)	#4); do not know
B is higher at the starting point.(3)	e.g., "[students would be thinking] B is higher at the starting point." (Questionnaire-L-#3, #4, & #7)	what x- and y-axis mean (See Interview Chinese –L-#2); careless mistakes (See Interview Chinese –L-#5).
	Correcting approaches	
Carefully read the graph again to understand the information given in the graph.(1)	e.g., "Ask students to carefully read the graph again to understand the information given in the problem." (Questionnaire Chinese-L-#9)	
Provide real-life examples to understand the relationship between distance, time and speed.(4)	e.g., "[I would] provide some interesting motions in real world, such as racing, for my students to understand the relationship between distance, time and speed." (Questionnaire Chinese-L-#5)	Make math learning interesting and meaningful. (See Interview Chinese-L-#3)
Teach and discuss the functional relationship between distance, speed and time (6)	e.g., "[I would] initiate a discussion on the relationship between distance, speed and time." (Questionnaire Chinese-L-#1)	Students' lower ability. (See Interview Chinese-L-#1 & #2)
Ask students to remember the formula of speed.(3)	e.g., "[I would] ask students to remember the formula of speed as V=D/T. (Questionnaire Chinese-L-#10)	

Note. Numbers in parentheses indicate the number of responses that fall into that category.

There are seven mathematical ideas or concepts mentioned in teachers' questionnaire responses in this group: the meaning of function graphs in real life; slope/rate of change in the graph and meaning; meaning of *x*-value and *y*-value on the graph; meaning of distance, speed and time and the relationship between them; definition of function; formula of speed as *V=D/T*; and translation between equations and graphs.

Nine of the fourteen teachers in this group mentioned that understanding the relationship between distance, speed and time was important. As Ms. Zheng wrote in her questionnaire response, "The key of this problem is an understanding of the relationship between distance, speed and time and their representations on the graph." (Questionnaire Chinese-L-#5) Mr. Wang had a similar response on the questionnaire, "[It is] about distance, time and speed." (Questionnaire Chinese-L-#6)

Seven teachers pointed out the importance of understanding the meaning of function graphs in real-life situations. As Ms. Zhou wrote in her questionnaire response,

> [Students need to] understand what a graph means in real life. A graph can represent a relationship between time point and temperature, a graph representing a relationship between the diameter and the area of a circle. Students need to make these connections. For this problem, it is about motion. (Questionnaire Chinese-L-#1)

Four of the teachers mentioned the importance of being familiar with the formula of speed. For example, Mr. Wang wrote in his questionnaire response, "[Students need to] know how to get speed from a problem. They are supposed to remember the formula V=D/T." (Questionnaire Chinese-L-#6)

Three teachers in this group mentioned that students need to understand the concept of slope or rate of change to correctly solve this problem. For example, Mr. Wu wrote on his questionnaire, "Slope is a very important concept to solve this problem. It is a comparison of slopes. Students need to understand what a slope means and how to determine it." (Questionnaire Chinese-L-#2)

Two teachers in this group wrote in their questionnaire responses that it was important to "understand the definition of function," (Questionnaire Chinese-L-#13) and two teacher wrote that "[students need to] understand the meaning of x-values on the graphs and the meaning of y-values on the graph." (Questionnaire Chinese-L-#10)

Additionally, one teacher wrote in her questionnaire response that, "[Students need to] be able to translate between equations and graphs of functions."(Questionnaire Chinese-L-#8)

When asked what they thought their students might be thinking when solving this problem, teachers in this group provided the following responses to the questionnaire. Two of the fourteen teachers wrote in their questionnaire response that, "[Students might be thinking] B is higher at t=2 on the graph which means B moves faster." (e.g., Questionnaire Chinese-L- #8 & #9)

Nine of the fourteen teachers in this group wrote in their questionnaire responses that "[students might be thinking] B is higher on the graph." (Questionnaire Chinese-L-#1 & #2)

Additionally, three teachers wrote in their questionnaire response that "[students would be thinking] B is higher at the starting point." (Questionnaire-L-#3, #4, & #7)

When asked why they thought their students might have this thinking, three types of explanations were found from the interview data in this group. Ms. Zhou, Ms. Zheng and Mr. Wang pointed to students' lack of ability of reading graphs on a coordinate plane. For example, Ms. Zhou explained in her interview,

> You know some of my kids…they really do not know how to read graphs on a coordinate plane. They cannot make meaning of the graph. They do not know how to analyze the information given in the graph to figure out the solution. Like this problem, this is actually a little tricky. Kids are very visual. Before they start analyzing what information this problem and what it asks, they use their 'eyes'…you know what I mean? That's why they see B is higher and moves faster…They do not know what a graph means. (Interview Chinese-L-#1)

Similarly, Mr. Wang explained in his interview,

> They are thinking B starts at a higher position so B moves faster and I think it is probably because kids do not know how to correctly read a graph. What does this graph mean? How does a graph reflect a relationship in a real-life situation? They are lack of this skill… (Interview Chinese-L-#4)

Mr. Wu, who pointed to the lack of familiarity with the *x*-axis and *y*-axis of a coordinate plane, said in his interview,

> Understand coordinate planes, uh, this is my requirement for my kids. It is difficult for my kids. They like drawing, like graphs, like visual stuff…but when it comes to coordinate plans which requires them to do math with, requires them to understand the meaning, they are 'lost'… Why do they think B is higher so B moves faster? They do not understand the *x*-axis mean. They do not understand

the *y*-axis mean. They should know a coordinate plane can represent a type of relationship. For example, this coordinate plane represents motions. All lines on the plane represent different motions. They may have the same speed or slope. They may not. But they share the same *x*-axis and *y*-axis. (Interview Chinese-L-#2)

In the interview with Ms. Feng, she pointed to the carelessness as the major reason behind this mistake. Ms. Feng said,

> This is their typical mistake, careless errors. I have told them many times: carefully read the information given in a problem, use that information to analyze and get a solution. In our guided practice, they do not have this problem because I analyze for them. But when it comes to tests which they need to deal with this by alone, they just forget. (Interview Chinese-L-#5)

When asked how they would help their students correct the mistake in the scenario, teachers in this group provided the following approaches according to their questionnaire data.

Ms. Feng, who pointed to the carelessness as the main reason behind this mistake, wrote that she would "ask students to carefully read the graph again to understand the information given in the problem." (Questionnaire Chinese-L-#9) She explained in her interview,

> I would ask them to read the graph and the problem again to get the information. Considering the time we spent on this type of problems, I do not think they are not able to understand the concept and the relationship. I do not think they read the problem carefully. They make all kinds of careless errors at this age. I would ask them to be more careful, careful, and careful… (Interview Chinese-L-#5)

Six of the fourteen teachers in this group mentioned that they would teach and discuss with class the relationship between distance, time and speed to correct this mistake. For example, Ms. Zhou wrote on her questionnaire, "[I would] initiate a discussion on the relationship between distance, speed and time." (Questionnaire Chinese-L-#1)

She explained this approach in the interview,

> …what if they do not understand the relationship between distance, time and speed and their representation on a graph…that is my concern…I think I would like to reteach this content even we did spend time on it in the past…I just do not have much confidence in my kids if they still remember it when it comes to problem-solving. Slope and steepness of a line is a difficult part for my kids especially when it comes to comparisons. I want them to relearn it and have some solid understanding in their head… (Interview Chinese-L-#1)

Mr. Wu, who chose a similar approach that emphasized an instruction on the relationship between speed, time and distance, explained in his interview,

> I doubt if they really understand…they are low…I would ask them to discuss the relationship between distance, time and speed in class. This content is tested a lot times on the high school entrance exams, in a more difficult way…I want them to really understand these concepts and their relationship as well as how to figure them out from a graph. When my kids go to the ninth grade and high school, motions become more complicated…For my eighth graders, at least I want them to have a solid understanding of uniform motions and their graphs on the coordinate plane. (Interview Chinese-L-#2)

Four teachers mentioned in their questionnaire responses that they would provide some real-life examples. For example, Ms. Zheng wrote, "[I would] provide some interesting motions in real world, such as racing, for my students to understand the relationship between distance, time and speed." (Questionnaire Chinese-L-#5)

She explained this approach in her interview,

> ...You can directly give them correct answers and show them the procedures. Kids can repeat your procedures without a real understanding. I want them to really understand what is happening in this graph, in this relationship. I want them to be involved, to be interested. They learn when they are interested. They learn when they make connections. I see real-life examples as important to hook up kids' interest. It makes sense to them. It is not only a math problem. It is real life, which has meaning. I will also ask them to come up with examples of motions, make up stories to describe the graph. I want them to be engaged… (Interview Chinese-L-#3)

Three teachers mentioned in their questionnaire response that to remember the formula of speed. For example, one teacher wrote, "[I would] ask students to remember the formula of speed as $V=D/T$." (Questionnaire Chinese-L-#10)

Four types of approaches of correcting this mistake were found in this group. Type 1: Ask students to carefully read the graph again to understand the information given in the graph. Type 2: Provide real-life examples to understand the relationship between distance, time and speed. Type 3: Teach and discuss with class the relationship between distance, speed and time. Type 4: Ask students to remember the formula of speed.

Curricular knowledge.

In this section, I provide a description of the Chinese lower SES middle school math teachers' curricular knowledge (see Table 27). Specifically, I first present these teachers' response to what instructional materials, textbooks, in particular, they use for teaching the topic of functions. The data source for this is the questionnaires collected from these teachers. I then present these teachers' explanations of how they use these instructional materials, textbooks, in particular, in the classrooms. The data source for this is the interviews. Lastly, I present these teachers' lateral and vertical curriculum knowledge. The data source for both the lateral and vertical curriculum knowledge is the interviews.

Table 27

Chinese Lower SES Teacher Group's Curricular Knowledge

Instructional materials.	Standards.(6)
	Renjiaoban Mathematics (11), *Huashidaban Mathematics* (4), and *Beijing Kegaiban Mathematics* (3)
	Outside materials.(7)
The use of textbooks.	Use some examples/problems in the book; complement problems from outside materials.
Functions and other subject matters.	Very little cooperation.
	Include some simple physics topics.
Functions and other topics in math.	The topic of functions as the most important one in students' math learning and math topics are connected.

According to teachers' questionnaire responses, nearly half of the teachers (six teachers) in the Chinese lower SES middle school math teachers group mentioned that

they included the *New Standards* in their instructional materials (see e.g., Questionnaire Chinese-L-#1 & #5). Eleven out of fourteen teachers in this group wrote on their questionnaires that they used the *Renjiaoban Mathematics* (see e.g., Questionnaire Chinese-L-#1 & #4). Four teachers wrote that they used both the *Renjiaoban Mathematics* and *Huashidaban Mathematics* which was published by the East China Normal University Press in class (see e.g., Questionnaire Chinese-L-#5 7 #7). Three teachers wrote that they used the *Beijing Kegaiban Mathematics* in class (see e.g., Questionnaire Chinese-L-#3). In addition, about half of the teachers in this group mentioned that they used outside materials to supplement their textbooks. Teachers' demonstration of how they used their textbooks pointed to the fact that they used some examples and problems in the book and supplemented some problems from outside materials (see e.g., Questionnaire Chinese-L-#2 & #5).

In the interviews, teachers suggested how they used their instructional materials in class as follows. For example, Ms. Zheng said in her interview,

> I select some examples and problems from the textbook and include them in my teaching. It is not easy for my kids to understand the problems in the book. One of my goals is to make sure that kids in my class are able to solve the problems in our textbook. These problems are usually basic-level problems. That's the first step toward the test… Important but not enough…I use outside materials to complement our textbook. Kids get more practice and more higher-level problems from outside materials. If they want to get more points on the high school entrance exam, they need to do a lot practice on higher-level problems… (Interview Chinese-L-#3)

Similarly, Ms. Feng explained,

> Yes, I use the textbook-definitions, examples and problems…textbook is important. I mean you have to put standards and textbook at hand when you prepare your lesson, right? That's the guide. Uh, but I do not think you should follow all the examples of the textbook. Some examples do make much sense to my kids. And more importantly, there are not enough problems there for kids to practice. I always give them problems from outside materials so that they can practice. And also, problems in the book are basic level or a little bit higher than basic. They need higher-level problems to practice in order to pass the exam. (Interview Chinese-L-#5)

Another explanation was found from Mr. Wu's interview,

> I do not use many examples from the book. It is because they do preview the content in the textbook before class and they are already familiar with the examples for that lesson. If I use the examples again in teaching, they will simply lose their interest. So I usually use examples from my own resources. I then ask them to do some practice using problems in the book. And I will also give them some higher-level problems to solve from my materials. (Interview Chinese-L-#2)

According to teachers' interviews in this group, when asked about the interdisciplinary work they had done, teachers suggested that they had incorporated some physics concepts into their teaching if necessary, but overall they had done very little interdisciplinary work. For example, Mr. Wu said,

> When I was teaching linear functions, I introduced the concept of distance and speed. I told them it was gonna be taught again in their physics class. Actually,

the physics teacher asked me before I taught this section, 'have you started teaching the linear system yet?' He needed me to teach that first, and he then can teach the velocity topic… honestly, I do not incorporate much from other disciplines. Sometimes when our math problems relate to some physics or chemistry topics, they do not ask kids to show that they understand those physics or chemistry concepts. What they ask for is that students understand the relationship between the quantities rather than those physics concepts per se. (Interview Chinese-L-#2)

In another interview, Ms. Zheng said,

I think physics is the discipline related to math quite often. I sometime would introduce them some physics concepts in my class. Uh, and physical formulas. But I don't communicate much with the physics teachers. There is a division between physics and math, I think…And we do not have the same requirement for kids. For example, in physics, they place a lot of emphases on the name of units during the calculation process. But in math, we do not ask kids to put the name of units during the calculation. Instead, we ask them to put that at the end. This conflicts with the requirement in the physics. Our physics teachers complain to us a couple of time about this 'cause kids are kind of confused and frequently forget to put units there. (Interview Chinese-L-#3)

In the interviews, when asked to describe the connections between functions (math) and other topics in math learning, teachers suggested that the topic of functions was the most important one in students' math learning and math topics were connected. For example, Mr. Wu said in his interview,

I think that function is built upon a lot of math knowledge points. It relates to equations, inequalities, expression, etc. The topic of functions is at the most important location of the math learning if we view all math knowledge as a connected network. It relates to everything in Algebra. And it also can relate to geometry or probability. Actually all the math topics are connected in secondary school. (Interview Chinese-L-#2)

In another interview, Ms. Zheng said,

I think it is our work to teach our kids how to connect the math knowledge they have learned. They need to know the relationships between those math topics… functions are the most important topic in Algebra, uh, and in students' entire math learning in school. They need to how functions relate to numbers, expressions, equations, graphs, and so forth. They need to know the math knowledge they learned in elementary school are related to what they are learning in middle school, and it also will relate to what they are going to learn in high school. I'd like to ask them to draw a math network chart. See how topics are connected to each other… (Interview Chinese-L-#3)

CHAPTER 5

CROSS-CASE RESULTS AND DISCUSSION

In Chapter 5, I provide cross-case results, discussion and concluding remarks for each research question (i.e. middle school math teachers' knowledge of instruction; teachers' knowledge of student thinking; and teachers' curricular knowledge) of the study. The research questions are put forward as reflective of the three key components – instructional knowledge, understanding of student thinking, and curricular knowledge - of Shulman's (1986, 1987) PCK model.

For each sub- research question of my research questions, I first provide table(s) and description to display the cross-case results of the four teacher groups. I then provide discussion for those cross-case results. The discussion is focused on capturing the differences and similarities among the four teacher groups, on the one hand; and the consistency with previous research findings, on the other. At the end of each sub-section, I provide a summary of findings for each sub-research question.

Research Question 1: Instructional Decisions

Research question 1 of this study is stated as follows,

What instructional decisions do the U.S. and Chinese higher/lower SES middle school mathematics teachers make to introduce the concept of function?

Three sub-research questions are put forward to answer this research question.

Instructional goals.

The first sub-research question of research question 1 is stated as follows,

What instruction goals do the U.S. and Chinese higher/lower SES middle school math teachers set for their introductory lesson of functions? What are the underlying reasons for their instructional decisions?

Cross-case results.

Table 28

Instructional Goals across Teacher Groups

U.S. higher SES (10)	U.S. lower SES (10)	Chinese higher SES (11)	Chinese lower SES (14)
Goal: Understand function definition as one input with exactly one output.(10) Explanation: Standards (the Common Core Standards).	Goal: Understand function as one input with exactly one output.(7) Explanation: Standards.	Goal: Understand function definition as one input with exactly one output.(6) Explanation: Standards and textbook; most difficult part is the definition.	Goal: Understand function definition as one input with exactly one output.(12) Explanation: Standards and textbooks.

Goal: Understand the concept of domain and range.(6) Explanation: Help understand the definition of function.	N/A	Goal: Understand range of x-value and range of y-value (domain and range).(6) Explanation: Make sense of a function; high school learning concern.	N/A
Goal: Understand and use the vertical line test.(6) Explanation: An easier way to identify functions; and one more perspective to define a function.	Goal: Understand the vertical line test.(3) Explanation: Help identify functions.	N/A	N/A
Goal: Understand graphical representations of functions.(4) Explanation: Help understand what a function is as one important form of	Goal: Understand graphical representations.(4) Explanation: Help understand what a function is as one important form of representation;	N/A	Goal: Understand the graphical representation of functions.(3) Explanation: Students is going to have difficulties in graphs.

representation.	student interest.		
Goal: Understand the concept of input and output.(2)	Goal: Understand Input/output.(4) Explanation: Foundation of learning functions –student ability.	N/A	N/A
Goal: Understand the concept of slope/rate of change.(1)	Goal: Understand the concept of rate of change.(1)	Goal: Understand the concept of slope/rate of change (1)	N/A
Goal: Identify linear functions.(1)	N/A	Goal: Understand linear functions.(1)	Goal: Understand what linear functions are.(3)
N/A	Goal: Understand the concept of variable.(2)	N/A	N/A
N/A	Goal: Understand coordinate plane.(1)	N/A	Goal: Understand the meaning of coordinate plane.(1)
N/A	Goal: Understand Independent/dependent variable.(1)	Goal: Understand Independent and dependent variable.(3)	Goal: Understand independent and dependent variable.(6) Explanation: Help understand a function; standards and textbook (and

			student ability).
N/A	N/A	Goal: Understand three forms of representations – equations, tables and graphs.(7) Explanation: Standards and textbooks; and understand a function thoroughly.	Goal: Understand three representations of functions.(5) Explanation: Standards and textbook.
N/A	N/A	Goal: Understand and write algebraic functions.(8) Explanation: The most important form to understand a function; high-school exam.	Goal: Understand the algebraic representations of functions.(5) Explanation: The most important representation; difficult for kids on tests.
N/A	N/A	Goal: Understand the concept of constant and variable.(5) Explanation: Textbook/sequence	Goal: Understand the concept of constant and variable.(11) Explanation: Sequence the

		the curriculum.	curriculum and textbook (and student ability).
N/A	N/A	Goal: Understand patterns, relations and function.(5) Explanation: Math is connected; most difficult part is finding the pattern.	N/A
N/A	N/A	Goal: Understand change and correspondence.(3)	Goal: Understand the concept of change.(2)

Note. Numbers in parenthesis indicate the number of responses that fall into that category.

As shown in Table 28, four instructional goals were frequently stated in the lesson plans of the U.S. higher SES middle school math teachers group in this study. First, all the ten teachers in this group wrote in their lesson plans that they expected their students to understand that a function is a rule that assigns to each input exactly one output in the introductory class (see Lesson plan U.S.-H-#4 & #8). As teachers explained in the interviews, they set this as a goal because this is "explicitly stated in the Common Core Standards" (see Interview U.S.-H-#1 Interview U.S.-H-#3 in Chapter 4). Second, six teachers wrote in their lesson plans that they expected their students to understand the concept of domain and range (e.g., Lesson plan U.S.-H-#2, #3). Teachers who included this goal explained in the interviews that "understanding domain and range helps

understand what a function is." (Interview U.S.-H-#1 & #2) Third, more than half (six) teachers wrote in their lesson plans that they expected their students to be able to apply the vertical line test in identifying functions (e.g., Lesson plan U.S.-H-#4). They explained this goal in the interview that the vertical line test is an easier way to identify functions (see Interview U.S-H-#3 & #4 in Chapter 4). on the one hand, and provides one more perspective to define a function (see Interview U.S.-H-#2), on the other. Fourth, almost half (four) of the teachers in this group wrote in their lesson plans that they expected their students to be able to understand (and use) graphical representations of functions in the class (e.g., Lesson plan U.S.-H-#5 & #6). According to their interviews, teachers set this goal mainly because graphs are visual and "looking at graphs is a good way to help student identify is that a function or not." (Interview U.S.-H-#4)

In the U.S. lower SES middle school math teacher group, four instructional goals were frequently mentioned in these teachers' lesson plans (see Table 5.1). The most frequently mentioned instructional goal in this group is that understanding that function is a rule which assigns each input exactly one output (e.g., Lesson plan U.S.-L-#4). This is required by the Standards (*The Common Core Standards*), according to teachers' explanations (see Interview U.S.-L-#1 & #4). Second, understanding the concept of input and output is another frequently stated instructional goal according to teachers' lesson plan data (e.g., Lesson plan U.S.-L-#4). As Ms. Iverson explained in the interview (see Interview U.S.-L-#4), understanding the concepts of input and output is the foundation of function learning and students may have difficulties in understanding these concepts when learning functions. Third, students are expected to understand how to draw function graphs and to be able to identify function based on graphs is another important goal

stated in teachers' lesson plans (see Lesson plan U.S.-L-#10). According to teachers' explanation, graphs helped students understand what a function is and graphs can also "hook up" students' interest in learning the topic of functions (e.g., see Interview U.S.-L-#3 & #6). Lastly, to be able to use the vertical line test to identify functions is a frequently mentioned instructional goal in teachers' introductory lesson plans (see Lesson plan U.S.-L-#3) since it is "easy" and convenient in identifying functions (and function graphs) (see Interview U.S.-L-#5).

In the Chinese higher SES middle school math teachers' group, there are five instructional goals which were frequently stated in their lesson plans. First, students were expected to under the function definition, that is, understanding that in a changing process, if there are two variables x and y, for every x there is a unique y (e.g., see Lesson plan Chinese-H-#3). The reasons for setting this goal, as teachers in the group explained, were twofold. Understanding the definition is required by the standards and textbooks, on the one hand, and it is actually difficult to really understand the definition of function (see Interview Chinese-H-#1 & #2). Second, students were expected to understand patterns, relations, and functions and their relationships, as stated in five teachers' lesson plans (e.g., see Lesson plan Chinese-H-#2). It is, based on teachers' explanations, because math is connected on the one hand (see Interview Chinese-H-#2); and one of the most difficult skills in math learning is finding the patterns, on the other (see Interview Chinese-H-#4). Third, students were expected to understand all the three forms of representations – algebraic, tabular and graphic representation (e.g., see Lesson plan Chinese-H-#9) since it is required by the *New Standards* and textbooks and it is important in helping understand a function thoroughly (see Interview Chinese –H-#3 & #4). More importantly, teachers

also mentioned in their lesson plans that students were expected to understand and use algebraic functions (equations) in solving math in this introductory lesson. It was actually the most frequently stated goal—mentioned by eight teachers in their lesson plans—in this group (see Lesson plan Chinese-H-#6 & #9). Algebraic functions, according to teachers' explanations, are the most important form of representation in understanding a function and it is important in the high school entrance exam (see Interview Chinese-H-#1, & #4). Fourth, understanding the concepts of constant and variable is a frequently mentioned instructional goal, as stated in five teachers' lesson plans (e.g., see Lesson plan Chinese-H-#4), since it is explicitly stated in the textbooks (see Interview Chinese-H-#3). Lastly, students were expected to understand domain (or the range of x-values) and range or (range of y-values), as stated in six teachers' lesson plans (e.g., see Lesson plan Chinese-H-#1 & #5). Without an understanding of domain, as teachers explained in the interviews (see Interview Chinese-H-#1 & #4), one cannot make sense of a function in real-life situations. Additionally, domain and range are important concepts in the high school math learning (see Interview Chinese-H-#4).

In the Chinese lower SES middle school math teachers' group, there are several instructional goals that were frequently mentioned according to teachers' lesson plan data. First, the most frequently mentioned goal is that students were expected to understand the definition of function (see Lesson plan Chinese-L-#3 & #7). That is, understand that in a changing process, if there are two variables x and y, for every x there is a unique y. The reason for understanding function definition, as teachers' explained in this group, was that it was explicitly stated in the New Standards and textbooks (see Interview Chinese-L-#1 & #4). Second, understanding the concepts of constant and variable and

understanding the concepts of independent and dependent variable are two frequently mentioned instructional goals in teachers' lesson plans (see Lesson plan Chinese-L-#3, #4, & #5). The reasons for setting understanding constant and variable as an important goal were similar to that of setting the goal of understanding the concepts of independent and dependent variable. According to teachers' explanations, when setting the goals, teachers were inclined to sequence the curriculum (the *New Standards*) and textbook, on the one hand (see Interview Chinese-L-#1, #2, & #4); and they also had concerns on their students' lower ability level which made it necessary for them to address these concepts at the introductory lesson (see Interview Chinese-L-#1 & #4). Third, students were expected to understand the three forms of representation of functions--algebraic, tabular and graphical representations--as stated in five teachers' lesson plans (e.g., see Lesson plan Chinese-L-#8) since it was explicitly stated in the *New Standards* and textbooks (see Interview Chinese-L-#3 & #5). Third, being able to use algebraic representation of functions in solving math problems is an important goal stated in teachers' lesson plans (e.g., see Lesson plan Chinese-L-#3, #11). As teachers explained (see Interview Chinese-L-#1 & #5), the algebraic form of representation was the most important representation of functions and it was difficult for kids on tests, especially on the high school entrance exam.

Discussion.

All the four teacher groups in this study pointed out understanding the definition of function as important instructional goal in their introductory lessons. From teachers' explanations, it is obvious that this goal is included primarily due to the requirements in the standards and textbooks in both the U.S. and China. Specifically, teachers from both

the U.S. higher and lower SES middle school teacher group in this study are more likely to describe this goal as "students are expected to understand that function is a rule that assigns to each input exactly one output," same as stated in the *Common Core Standards*. Students are expected to understand the definition from an input/output perspective. A little differently, teachers from both the Chinese higher and lower SES middle school teacher groups in this study are more likely to describe this goal as "students are expected to understand the function definition as "in a changing process, if there are two variables, for every x there is a unique y," explicitly stated in the *New Standards* and in the textbooks such as *Renjiaoban*. Students are expected to understand the definition from a "change-and-correspondence" perspective. Although describing the definition of function from two different perspectives, all the teacher groups in this study actually share commonalities in teaching the definition of function in the introductory lesson. First of all, despite cultural differences, using a definition of function to introduce the topic is a common method. It is consistent to Sfard's contention (1992) that structural teaching was a common way of teaching, in which new concepts begin their life in the classroom as ready-made objects. The way to introduce to students function as a well-defined concept in the introductory lesson is an obviously structural approach of introducing the topic of functions. The structural approach is, from the present study, still a common approach of beginning a new topic across countries and SES. In addition, the function definition – the well-defined concept, as described in their lesson plans, serves also as the rule for identifying functions in the future lessons for all the four teacher groups in this study. Second, correspondence is emphasized in both perspectives of understanding function definition - the correspondence between input and output and the correspondence two

variables x and y. Thinking of function from correspondence can promote student conceptions of a function as a single object or as a process or both (Schwingendorf et al., 1992). Although students' conceptions of functions are not explored in the present study, the possibilities that these teachers bringing in through defining what a function is are still present.

Being able to understand and use representations of functions is another important instructional goal of the introductory lesson in all the four teacher groups' lesson plans in this study. The forms of representation, however, emphasized in this instructional goal are different across countries. Specifically, both U.S. higher and lower SES middle school math teacher groups explicitly state their goal that students are expected to understand and use the graphical form of representation of functions as well as to apply the vertical line test in identifying function graphs. Differently, both the Chinese higher and lower SES middle school teacher groups explicitly emphasize that students are expected to understand and use the algebraic form of representation in solving math problems in the introductory lesson. This difference found in the present study is congruent to the previous finding by Cai and Wang's (2006) that Chinese teachers are more likely to use algebraic representation. In addition, according to teachers' own explanations, teachers' expectations on students' use of representations come from how they themselves understand the importance of different forms of representation of functions. In this study, the U.S. teachers interviewed state the importance of the representation of functions in understanding the concept of function. In contrast, the Chinese teachers interviewed state the importance of the algebraic representation both in understanding what a function is and on the high school entrance exam.

Moreover, the U.S. higher SES middle school math teacher group state in their instructional goals that they expect their students to understand the concept of domain and range even though this is required in the high school math rather than in the middle school math. Similarly, the Chinese higher SES middle school math teacher groups state in their instructional goals that they expect their students to understand the domain or the range of *x*-value and the range or the range of *y*-value even though this is required in the high school math rather than in the middle school math. Teachers in both groups explain that understanding the concept of domain and range is important in helping students understand what a function is both in math language and in real-life situations as well as its role in high school math learning, which to a certain degree reflects teachers' confidence in their students' ability of understanding a concept or a pair of terms which should be formally introduced in high school. It then can be argued that teachers in both higher SES middle school math teacher groups in the present study are more likely to come up with higher expectations for their students. Contradictorily, the U.S. lower SES middle school math teacher group state in their instructional goals that students are expected to understand the concept of input/output in this introductory lesson. The Chinese lower SES middle school math teacher group state in their instructional goals that students are expected to understand constant and variable as well as independent and dependent variable in the introductory lesson. According to teachers' explanations from these two groups, these are basic concepts that are closely related to the concept of function. Teachers in the two groups explain that they need to spend a quite large amount of class time to teach these basic concepts in the introductory lesson even though these concepts are actually taught to their students before because their students are relatively

lower compared to higher level students. The explanations reflect these teachers' concern of their students' ability of understanding of related knowledge of functions. It then can be argued that teachers in both lower SES middle school math teacher groups in the present study have the inclination to set lower-level goal for their students. From the discussion above, it is arguably obvious that teachers from higher SES middle schools tend to have higher expectations for their students in the introductory class in this study. This difference is largely due to teachers' different confidence levels and beliefs in their students' ability across SES.

Summary.

A review of the sub-research question: What instruction goals do the U.S. and Chinese higher/lower SES middle school math teachers set for their introductory lesson of functions? What are the underlying reasons for their instructional goals?

Table 29

Summary for Differences/Similarities and Underlying Reasons in Teachers' Instructional Goals

Similarity	Understand the definition of function and multiple forms of representations of functions in the introductory lesson. Underlying reason: Standards/sequence the curriculum.	
Difference	Teachers from higher SES schools (U.S. and China): Understand higher-level concepts, such as domain and range. Underlying reason: Higher expectations for students who	Teachers from lower SES schools (U.S. and China): Understand lower-level concepts, such as input/output and constant/variable. Underlying reason: Lower expectations for students who

	were at higher ability level.	were at a lower ability level.

See Table 29 for a summary of the major findings from the discussion above. Understanding the definition of function and multiple forms of representations of functions are two major instructional goals in the introductory lesson in all the four teacher groups of this study. Teachers do not only share commonalities in setting the major instructional goals for the introductory lesson, but also share the similar reason for setting these goals – following the standards (the *Common Core Standards* in the U.S. and the *New Standards* in China).

The major difference regarding teachers' instructional goals in this study is found between the higher-socioeconomic status middle school math teachers groups and lower-socioeconomic status middle school math teachers groups. In other words, it is a cross-SES difference rather than a cross-country difference found in the instructional goals. While teachers from higher SES schools expect their students to understand the concepts of domain and range which is required in high school, teachers from lower SES schools are more likely to expect their students to understand basic concepts, such as input and output and variable, in the introductory lesson of functions. Students' ability level, based on teachers' explanations, is the major concern when they set these goals for their students. In this study, teachers from higher SES middle schools are more likely to have higher expectations for their students who were believed to be at a higher ability level, whereas teachers from lower SES middle schools are more likely to have lower expectations for their students who were believed to be at a lower ability level.

Mathematical tasks.

The second sub-research question of research question 1 is stated as follows,

What mathematical tasks do the U.S. and Chinese higher/lower SES middle school math teachers construct for their introductory lesson of functions? What are their underlying reasons for these mathematical tasks?

Table 30

Construction of Mathematical Tasks across Teacher Groups

U.S. Higher SES	U.S. lower SES	Chinese higher SES	Chinese Lower SES
Memorization			
N/A	N/A	N/A	Response: 4 Explanations: Student's lower ability level; students cannot really understand concepts at this age; lack of prior knowledge.
Procedures without connection to concepts/meaning/understanding			
Response: 2	Response: 5 Explanations: Warm-up activities as transitioning into math thinking; make up for prior knowledge.	Response: 1	N/A
Procedures with connection to concepts/meaning/understanding			
Response: 14	Response: 10	Response: 22	Response: 25

Explanations: Common core transition; make meaning of what you are learning; kids are smart.	Explanations: Common Core adoption; make introductory class fun and relevant	Explanations: New standards and high school entrance exam; Knowing procedures is not equal to a real understanding.	Explanations: Consider the new standards challenge and high school entrance exam.
		Doing mathematics	
Response: 4	Response: 2	Response: 4	N/A
Explanations: Belief in kids' ability.	Explanations: Challenge my kids.	Explanations: High school entrance exam; student ability is higher and they can do math.	

Cross-case results.

As shown in Table 30, in the U.S. higher SES middle school math teachers group, eighteen out of the twenty mathematical tasks teachers constructed for the introductory lesson were at a higher cognitive demand level. Only two mathematical tasks were procedures without connection to concepts, understanding or meaning, which were at the lower level of cognitive demand (e.g., see Lesson plan U.S.-H-#9). Teachers who were interviewed from this group did not construct any task at this level. Among those eighteen mathematical tasks at the higher cognitive demand level, fourteen of them were those procedures with connections to concepts, understanding, or meaning (see Lesson plan U.S.-H-#7, #8, & #10). Teachers' reasons for constructing these mathematical tasks, from their explanations, can be summarized as first, adapting into the challenge in the

Common Core Standards (see Interview U.S.-H-#1, #2, & #4); second, making meaning of math learning for students through these mathematical tasks (see Interview- U.S.-H-#1, #2, & #4); and third, beliefs in students' ability to complete the tasks (see Interview U.S.-H-#3). Four of these eighteen higher level mathematical tasks were "doing mathematics" tasks (e.g., see Lesson plan U.S.-H-#4). Teachers constructed these tasks primarily because they had strong confidence in students' ability and potentials according to their explanations (see Interview U.S.-H-#4).

In the U.S lower SES middle school math teachers group, ten out of the seventeen mathematical tasks teachers constructed for the introductory lesson were those procedures with connection to concepts, understanding or meaning, which were at a higher cognitive demand level (e.g., see Lesson plan U.S.-L-#4, #5, & #7). Teachers' explanations for these mathematical tasks were twofold. On the one hand, it was a way to address the challenge as a result of the Common Core adoption in their schools (see Interview U.S.-L-#2); and on the other hand, using tasks with connection to understanding and meaning made introductory class fun and relevant to their students (see Interview U.S.-L #1 & #5). Five mathematical tasks were those procedures without connections to concepts, understanding or meaning (see Lesson plan U.S.-L-#2 & #3). Teachers in this group usually constructed these mathematical tasks, as they explained, to serve as warm-up activities in order to keep students focused on math thinking (see Interview U.S.-L-#4 & #5), or to make up for students' prior knowledge (Interview U.S.-L-#1 & #5). Two mathematical tasks were "doing-mathematics" tasks which were at the highest cognitive demand level (see Lesson plan U.S.-L-#6). The reason of constructing this type of task, as Ms. Haley (see Interview U.S.-L-#3) explained, was teachers'

inclination to challenge their students even if students might feel uncomfortable considering their lower ability level.

In the Chinese higher SES middle school math teachers group in this study, twenty-two out of the twenty-seven mathematical tasks teachers constructed for the introductory lessons were those procedures with connections to concepts, understanding or meaning, at a higher cognitive demand level (see Lesson plan Chinese-H-#1, #3, #7, & #10). According to teachers' explanations in this group, on the one hand, these teachers needed to adapt into the changes in the *New Standards* and the high school entrance exam after the curriculum reform (see Interview Chinese-H-#1, #2, #3, & #4); and on the other hand, in their opinion, a real understanding is not equal to know simply the procedures (see Interview Chinese-H-#2 & #4). Four "doing mathematics" tasks, of a higher cognitive demand level, were constructed by these teachers (e.g., see Lesson plan Chinese-H-#3). The reasons for constructing this type of task were twofold. As Mr. Qian and Mr. Sun explained in the interviews (see Interview Chinese-H-#2 & #3), the high-demanding problems related to the topic of functions were frequently tested on the high school entrance exams on the one hand; and teachers had confidence in their students ability of dealing with these high-demanding mathematical tasks, on the other. Only one mathematical task was found at the lower-level of cognitive demand, that is, procedures without connections to concepts, understanding or meaning (see Lesson plan Chinese-H-#7), but no teacher interviewed in this group constructed this type of mathematical tasks.

Among the Chinese lower SES middle school math teachers group, twenty-five out of the twenty-nine mathematical tasks teachers constructed for the introductory lesson were those procedures with connections to concepts, understanding or meaning, a higher

cognitive demand level (e.g., see Lesson plan Chinese-L-#3, #5, & #7). According to teachers' explanations (see Interview Chinese-L-#3, #4, & #5), the main reason teachers constructed this type of mathematical tasks was their consideration of the *New Standards* challenge and of the high school entrance exam challenge after the latest curriculum reform. Four out of the twenty-nine mathematical tasks constructed in the introductory lesson were the memorization type of task, the lowest cognitive demand level (e.g., see Lesson plan Chinese-L-#4 & #5). The reasons for these mathematical tasks, based on teachers' explanations in the interviews, were threefold. First, their students were at a lower ability level (see Interview Chinese-L-#1 & #3); second, students would not be able to really understand concept at this age (see Interview Chinese-L-#1, #2, & #3); and third, students lacked prior knowledge in learning a new topic (see Interview Chinese-L-#3). It was necessary for teachers in this group to ask students to memorize or recall knowledge.

Discussion.

According to the perspective of Stein and her colleagues (Henningsen & Stein, 1997; Stein & Lane, 1996; Stein et al., 1996, 2000), mathematical instructional tasks can be classified into four categories based on their cognitive demand level. Lower-level tasks include memorization and procedure without connection to understanding, meaning, or concepts, whereas higher-level tasks include procedures with connection to understanding, meaning, or concepts and doing-mathematics tasks. In the present study, most frequently constructed mathematical tasks in each of the four teacher groups are those procedures with connection to understanding, meaning, or concepts – higher-demand-level tasks. Teachers' explanations of constructing this type of tasks point to

their inclinations to address the challenges brought by the new standards – the *Common Core Standards* in the U.S. and the *New Math Standards* in China. For Chinese teachers in this study, they also need to address the new challenges in the corresponding tests, the high school entrance exam in particular, which ask them to raise the cognitive demand level for students. According to teachers' explanations in both countries, the implementation of the new standards raises the cognitive demand level for students to solve math problems as well as asks for the real understanding. It is necessary for teachers in both countries to construct mathematical instructional tasks with connection to understanding, meaning or concepts instead of with only procedures or calculations. This indicates that teachers in this study, across countries and SES, tend to construct cognitive demand tasks for their students, which supports Huang and Cai's (2010) argument that the U.S. and Chinese teachers are both likely to present their students with cognitively demanding tasks. Additionally, teachers' desire to address the new challenges in the new standards in both countries may also reflect the positive influence of curriculum reforms (including both standards and tests) in both countries, similar to Huang and Cai's (2010) findings in their research.

Additionally, compared to their counterparts from the lower SES middle school teacher groups, teachers from the higher SES middle school teacher groups are more likely to present their students with the "doing-mathematics" type of mathematical instructional tasks, the highest-demand-level tasks in this study. Teachers interviewed in both the U.S. and Chinese higher SES middle school math teacher groups state their confidence in their students' ability in solving this type of tasks when asked to explain why to construct the tasks. In contrast, compared to their counterparts from the higher

SES middle school teacher groups, teachers from the lower SES middle school teacher groups are more likely to present their students with the mathematical tasks with lower demand level. Specifically, the U.S. lower SES teacher group most frequently presents the procedures without connection to meaning, understanding or the concepts and the Chinese lower SES teacher group is the only one presenting the memorization type of tasks. When asked to explain their reasons of constructing these tasks, teachers in both groups point to their concern on students including students' ability level and prior knowledge. Teachers' decisions on selecting or designing a particular mathematical instructional task must take students into considerations (Stein et al., 2000). The finding in this study supports this argument that teachers are very likely to consider their students' ability level and prior knowledge when deciding the cognitive demand level and types of the mathematical instructional tasks.

Summary.

A review of the sub-research question: What mathematical tasks do the U.S. and Chinese higher/lower SES middle school math teachers construct for their introductory lesson of functions? What are their underlying reasons for these mathematical tasks?

Table 31

Summary for Differences/Similarities in Teachers' Construction of Mathematical Tasks

Similarity	Teachers tend to construct mathematical tasks at a higher-level of cognitive demand-procedures with connection to concepts, meaning or understanding. Underlying reason: The implementation of the new standards raises the cognitive demand level for students to solve math problems in both U.S. and China.

Difference	Teachers from higher SES schools are more likely to construct most cognitive demanding tasks--"doing mathematics" tasks--for their students.	Teachers from lower SES schools are less likely to construct most cognitive demanding tasks--"doing mathematics" tasks--for their students.
	Underlying reason: Teachers' confidence in their students' ability in solving this type of tasks.	Underlying reason: Teachers' concerns about students' lower ability level and prior knowledge.

As shown in Table 31, teachers in all the four groups are inclined to construct higher cognitive-demand-level mathematical tasks which are connected to understanding, meaning or concepts for their students in the introductory class, because the implementation of the new standards in both the U.S. and China raises the cognitive demand level for students to solve math problems. It may reflect the positive influence of curriculum reforms in both the U.S. and China (see also Huang & Cai, 2010).

Teachers from the higher SES middle schools, however, are more likely to construct the most cognitive demanding mathematical tasks-"doing mathematics" tasks, compared to their counterparts in the lower SES middle school of this study. As discussed above, students' ability level is teachers' major concern when it comes to the decision of constructing the most cognitive demanding mathematical instructional tasks (see also Stein et al., 2000). In this study, while teachers from the higher SES middle schools have stronger confidence in their students' ability of solving high level tasks, teachers from the lower SES middle schools lack this confidence considering their students' lower ability level.

Representations of functions.

The third sub-research question of research question 1 is stated as follows,

What representations of functions do the U.S. and Chinese higher/lower SES middle school math teachers use to introduce the concept of function? What are the underlying reasons for their use of the representations?

Cross-case results.

Table 32

Use of Representations of Functions across Teacher Groups

	U.S. higher SES (10)	U.S. lower SES (10)	Chinese higher SES (10)	Chinese lower SES (14)
Equations only	N/A	N/A	Response: 3	Response: 3
Graphs only	Response: 1	N/A	N/A	N/A
Equations and tables	N/A	N/A	Response: 3	Response: 5
Equations and graphs	Response: 3	N/A	N/A	Response: 3
Tables and graphs	Response: 3	Response: 4	N/A	N/A
Tables, graphs and equations	Response: 1	Response: 1	Response: 4	Response: 5
Tables, graphs, equations, verbal	N/A	Response: 1	Response: 1	N/A
Graph, table, verbal	Response: 1	Response: 1	N/A	N/A
Graph and ordered pairs	Response: 1	N/A	N/A	N/A
Ordered pairs, graphs and tables	N/A	Response: 3	N/A	N/A

Teachers' attitudes toward representations
U.S. higher SES: Graphs are important in understanding functions; kids like visual and graphs; and graphs connect to their prior knowledge.
U.S. lower SES: Graphs and tables are important in understanding functions; and kids like visual, tangible things.
Chinese higher SES: Equation, table, and graph are three forms of representations of functions; equation is the most important form of representations as determining the nature of a function and as appearing in high school entrance exam; equations connect math knowledge points; and it is important to understand translation between equation and graph.
Chinese lower SES: Equation, table and graph are three forms of representations; equation is the most important form of representations of functions; and tabular and graphical representations are important in understanding functions.

As shown in Table 32, in the U.S. higher SES middle school math teachers group in this study, graphical representations were used in each of teachers' introductory lesson plans (see e.g., Lesson plan U.S.-H-#3 & #5). Graphs were frequently used with other forms of representations, such as equations and tables, in most of the introductory lesson plans. Nine of the ten teachers used a combination of graphs and other form(s) of representations in their lesson plans. As teachers' explained, being able to use multiple representations is required in the *Common Core Standards* (see e.g., Interview U.S.-H-#1). According to teachers' interviews of this group, two ideas were identified in teachers' explanations of their attitudes towards using representation of functions. On the one hand, every teacher interviewed mentioned that it was important to use graphical representations of functions because first of all, graphs were important in helping students understand functions (see Interview U.S.-H-#1); second, students liked drawing

graphs and visual things (see Interview U.S.-H-#1, #2, & #4); and third, using graphs can connect to students' prior knowledge (see Interview U.S.-H-#4). On the other hand, in these teachers' opinion (see Interview U.S.-H-#1 7 #3), it was important to be able to convert between graphic and other representations of functions for its important role in understanding a function and its difficulty as a math skill in solving related math problems.

In the U.S. lower SES middle school math teachers group in this study, graphic and tabular representations were used in each of the introductory lesson plans. Additionally, graphs and tables were used with other forms of representations, such as equations and verbal, in more than half of the introductory lesson plans. The use of multiple representations is encouraged since it is explicitly stated in the *Common Core Standards,* as Mr. Jordon explained (see e.g., Interview U.S.-L-#5). Based on teachers' interview data, two ideas were identified in teachers' explanations of their attitudes towards using representation of functions. First, as Ms. Fenning, Ms. Gerold and Mr. Jordon explained in the interviews (see Interview U.S.-L-#1,#2, &#5), tabular and graphical representations, especially graphical representations, were important in the introductory lessons. It is because on the one hand, graphs and tables can help students understand functions (see Interview U.S.-L-#1 & #2); and on the other hand, kids liked drawing graphs which were visual, tangible to them (see Interview U.S.-L-#5). Second, as Ms. Iverson and Ms. Gerold explained in their interviews (see Interview U.S.-L-#2 & #4), it was not necessary to emphasize algebraic functions in the first few classes because in teachers' experience, algebraic representations were too abstract to be understood by

students on the one hand; and not every function came with an algebraic representation, on the other.

In the Chinese higher SES middle school math teachers group in this study, algebraic representations were used in each of the introductory lesson plans (see e.g., Lesson plan Chinese-H-#5). In addition, algebraic representations were used with other forms of representations, such as graphs and tables, in most of the introductory lesson plans. Eight of the eleven teachers in this group used a combination of algebraic and other form(s) of representations. According to teachers' interviews in this group, three ideas were identified from teachers' explanations of their attitudes towards using representations of functions. First, algebraic, tabular and graphical representations were three forms of representations of functions that these teachers encouraged students to understand in the first lesson (see e.g., Interview Chinese-H-#1 & #2). According to Ms. Zhao's and Mr. Qian's explanations (see Interview Chinese –H-#1, #2), it was because these three forms were explicitly stated in the Standards and they would help students understand what a function is from different perspectives. Second, all the teachers who were interviewed stated that algebraic representations were the most important representations of functions (see Interview Chinese-H-#1, #2, #3, & #4). Algebraic representations were important because first, the algebraic form or the equation of a function determined all the features of a function (see Interview Chines-H-#1); second, algebraic functions appeared frequently on the high school entrance exams (see Interview Chinese-H-#3); and third, equations stayed at the center of math learning and connected students' prior knowledge (see Interview Chinese-H-#2, #3, & #4). Lastly, as Mr. Qian and Ms. Li explained in their interview (see Interview Chinese-H-#3, #4), it was

important for students to know how to translate between algebraic representations and other forms of representations, such as graphical representation, because it was a difficult math skill for middle school students and it was important in solving particular math problems on the high school entrance exam.

In the Chinese lower SES middle school math teachers group in this study, algebraic representations were used in each of the introductory lesson plans (see Lesson plan Chinese-L-#1 & #6). In addition, algebraic representations were used with other forms of representations, such as graphs and tables, in most of the introductory lesson plans. Eleven out of the fourteen teachers used a combination of equations and other form(s) of representations in their lesson plans. Based on teachers' interview data, three ideas were identified from teachers' explanations of their attitudes towards using representations of functions in this group. First, as Ms. Zhou and Ms. Zheng (see Interview Chinese-L-#1 & #3) explained in the interviews, students were expected to use algebraic, tabular and graphical representations of functions, three forms that were explicitly stated in the *New Standards* and textbooks. Second, as Ms. Zhou and Ms. Feng (see Interview Chinese-L-#1 & #5) explained, algebraic representations were the most important representations of functions which students were expected to understand because it was difficult for kids to use the algebraic representations on the high school entrance exam based on teachers' experience. Third, as Mr. Wu and Ms. Zheng explained in their interviews (see Interview Chinese-L-#2 & #3), tabular and graphical representations were important in helping students understand what a function is because of their visual features.

Discussion.

It is noteworthy that teachers in all the four groups tend to use multiple representations rather than one single representation in teaching the introductory lesson of functions. In the U.S. higher SES middle school math teacher group, graphs and tables or graphs and algebraic functions are frequently used together in teachers' lesson plans. Similarly, in the U.S. lower SES middle school math teacher group, graphs and tables (and mappings) are frequently used together in teachers' lesson plans. In the Chinese higher SES middle school math teacher group, algebraic functions and graphs or algebraic functions and tables are frequently used together in teachers' lesson plans. Similarly, in the Chinese lower SES middle school math teacher group, algebraic functions and tables or algebraic functions, tables and graphs are frequently used together in teachers' lesson plans. The use of multiple representations provides multiple perspectives for students to understand functions (Moschkovich et al., 1993). It seems that teachers in all the four groups of this study recognize the need to present students with multiple representations in introducing the new topic, a finding that is different from Friedlander and Tabach's argument (2001) that teachers do not often utilize multiple representations. As teachers explained in their interviews, students are expected to be able to use multiple representations of functions according to the requirement in the standards (both the *Common Core Standards* in the U.S. and the *New Standards* in China). Following what is explicitly stated in the standards is the major reason for the teachers of this study to use multiple representations in class.

However, teachers' attitudes toward each form of representations vary across countries in this study. From the description in the last section, it is obvious that teachers

in the U.S. higher SES middle school math teacher group use the graphical representation of functions in their lesson plans most frequently and teachers in the U.S. lower SES middle school math teacher group most frequently use graphical (and tabular) representations of functions in their lesson plans. These two teacher groups share commonalities with respect to teachers' attitudes toward the importance of using graphical representation in learning functions. Teachers in both the U.S. higher and lower SES middle school math teacher groups are more likely to use graphical representation of functions because on the one hand, it is important in helping students understand the concept of function; and it is easier for students at this age to understand graphs for their visual and tangible features, on the other. This is congruent to the previous research finding that it is more effective to teach lower-grade students function problems using graphs and tables since these are more intuitive and tangible for young students (Bell & Janvier, 1981; Moore-Rossuo & Golzy, 2005).

In contrast, teachers in both the Chinese higher and lower SES middle school math teacher groups use the algebraic representation of functions in their lesson plans most frequently. This indicates Chinese teachers' prioritization of algebraic representation in teaching and their inclination to encourage students to use the technical or mathematical language in solving problems in this study. Teachers from these two groups emphasize the role of algebraic functions apparently because of their presence in the high school entrance exam. It is obvious that high school entrance exam plays an important part in guiding teachers' use of and attitudes toward representation of functions. This is similar to the explanation that Chinese teachers provide for the construction of mathematical tasks in the last section. Additionally, teachers from the Chinese higher

SES middle school math teacher group also provide two more reasons for using algebraic representation of functions. One is teachers' understanding that algebraic functions can show all the features of functions as math models. This reflects Friedlander & Taback's (2001) argument that the algebraic representation of functions is the most effective and concise one in the presentation of mathematical models. The other is teachers' understanding of the role of algebraic functions in connecting students' prior and future knowledge, which resonates with the advocacy that it is important to learn functions in middle school because it builds the foundation of secondary school math (Cooney et al., 1996).

The cross-country difference on the use of representation of function found in the present study supports the idea that the U.S. teachers are more likely to encourage the use of graphical (and tabular) representation of functions whereas the Chinese teachers are more likely to encourage the use of algebraic representation of functions, which is consistent with previous research findings (e.g., Cai & Wang, 2006; Mesiti & Clarke, 2010).

Another important finding about teachers' attitudes toward the use of representation of functions is that teachers in both the U.S. and Chinese higher SES middle school teacher groups in this study explicitly state the importance of knowing how to translate among different forms of representations. Specifically, teachers in the U.S. higher SES middle school teacher group state that it is important for their students to understand the translation between graphical representation and other forms of representations, whereas teachers in the Chinese higher SES middle school teacher group state that it is important for their students to understand the translation between algebraic

representation and other forms of representations. From the discussion above, it is not surprising to see the U.S. higher SES teacher group place more value on the graphical representation while the Chinese higher SES teacher group place more emphasis on the algebraic representation in translations among representations of functions. What needs to be noticed here, however, is that teachers in both the higher SES teacher groups in the two countries have the awareness of training their students' ability of translating one form of representations to another. As Janvier (1985) contends, teaching functions can be very difficult because it involves a variety of representations and the translations among these representations. Teachers' explanations in both groups point to knowing how to translate among different forms of representation being an important and difficult skill for kids to grasp, which echoes Janvier's contention. Here again, the Chinese higher SES teacher group also points to the high school entrance exam as an important reason for understanding the translations among representations of functions, which is consistent to the finding above that tests play a significant part in the Chinese teachers' instruction in this study.

Summary.

A review of the sub-research question: What representations of functions do the U.S. and Chinese higher/lower SES middle school math teachers use to introduce the concept of function? What are the underlying reasons for their use of the representations of functions?

Table 33

Summary for Differences/Similarities in Teachers' Use of Representations of Functions

Similarity	Teachers in all the four groups tend to use multiple representations

	rather than one single representation in teaching the introductory lesson of functions. Underlying reason: Standards.	
Differences	U.S and Chinese higher SES: Emphasize the translation among different forms of representations. Underlying reason: students' higher ability level; and the high school entrance exam in China.	U.S. and Chinese lower SES: Did not emphasize the translation among different of representations. Underlying reason: students' lower ability level.
	U.S. higher and lower SES: Graphic representation. Underlying reason: Teachers' understanding of the role of graphic representation- tangible and visual.	Chinese higher and Lower SES: Algebraic representation. Underlying reason: Teachers' understanding of the role of algebraic representation – technical and effective mathematical language; and high school entrance exam.

As shown in Table 33, teachers in all the four groups share the commonality that they tend to use multiple representations rather than one single representation of functions in teaching the introductory lesson according to their lesson plan data, which is different from Friedlander and Tabach's (2001) argument that teachers did not often utilize multiple representations. Based on teachers' explanations in their interviews, being able to use multiple representations of functions is required by the standards (both the *Common Core Standards* in the U.S. and the *New Standards* in China), which encourages teachers to use multiple representations in class.

Two major differences were found in the four teacher groups regarding their attitudes toward the use of representations based on their interviews. One is a cross-SES difference that teachers from the U.S. and Chinese higher SES middle schools tended to emphasize the translation among different forms of representations of functions, whereas teachers from the U.S. and Chinese lower SES middle schools did not. This difference, as teachers explained, is a result of teachers' different expectations for their students. Translation among different forms of representation, as a difficult skill for kids to grasp, is more likely to be emphasized by the teachers from the higher SES middle schools of this study considering students' overall higher ability level in these schools.

The other difference is a cross-country difference - the U.S. teachers of this study are more likely to encourage the use of graphical representation of functions whereas the Chinese teachers are more likely to encourage the use of algebraic representation of functions - which is consistent with previous research findings (e.g., Cai & Wang, 2006; Mesiti & Clarke, 2010). This difference comes from teachers' different understanding of the role of representations of functions in the U.S. and China. While the U.S. teachers of this study are more likely to place higher value on the visual and tangible features of graphs in helping students understand the concept of function (see also Bell & Janvier, 1981; Moore-Rossuo & Golzy, 2005), the Chinese teachers of this study are more likely to place more emphasis on algebraic representation – a more technical (abstract) and effective mathematical language- in solving math problems (see also Cai & Wang, 2006; Friedlander & Taback, 2001). In addition, In China, the high frequency of algebraic functions appearing in the high school entrance exams encourages teachers' use of algebraic functions in class.

Concluding remarks for research question 1.

What instructional decisions do the U.S. and Chinese higher/lower SES middle school mathematics teachers make to introduce the concept of function? What are the underlying reasons for their instructional decisions?

I find that teachers of this study share quite a few similarities regarding their instructional decisions on the introductory lesson of functions (see Figure 5.1). First, they tend to set up the similar major instructional goal of understanding the definition of function in the first lesson; second, they are more likely to construct cognitive demanding mathematical tasks which are connected to concepts, understanding or meaning; and lastly, they all encourage the use of multiple representations in the introductory lesson. Standards play a significant role in all these instructional decisions. "Addressing the standards (i.e. the *Arizona Common Core Standards* and the *New Standards* in Beijing)" becomes one major reason behind teachers' instructional decisions. It may reflect the values that teachers place on the standards, indicating that standards-based instruction prevails in the classrooms of both the U.S. and China. It may also reflect the positive influence of the continuing efforts on the curriculum reform in both countries. Teachers of this study, across countries and SES, are inclined to address the new challenges in the new standards and curriculum.

I also find quite a few differences between teachers from the higher socio-economic status middle schools and teachers from the lower socio-economic status middle schools regarding their instructional decisions. Overall, teachers from the higher SES schools are more likely to emphasize higher-cognitive-level concepts, construct the most cognitively demanding mathematical tasks, and expect more difficult skills of using

representations in their classrooms, compared to their counterparts from the lower SES schools. Consideration on students' ability level plays an important part in these cross-SES differences in this study. Students in the higher SES schools of this study are believed to have higher math ability level; therefore they are expected to be able to understand more difficult concepts and complete higher cognitive-demand-level math tasks or problems, compared to their counterparts in the lower SES schools of this study.

The major cross-country difference I find regarding teachers' instructional decisions in the introductory lesson of functions lies in teachers' use of representations of functions. While the U.S. teachers of this study tend to place more values on the intuitive and tangible features of the graphic representations, the Chinese teachers of this study tend to place more emphasis on the technical or abstract feature of the algebraic representations of functions in their introductory classes. It may reflect teachers' own understanding of the topic of functions. Teachers' understanding of a particular content determines, to a large degree, their teaching of that content. It may also reflect teachers' beliefs in math learning. While the U.S. teachers of this study are inclined to rely on more concrete, visual and tangible "language" in math teaching and learning, the Chinese teachers of this study are inclined to rely on more technical, abstract and mathematical "language" in math teaching learning. In addition, testing (i.e. high school entrance exam) plays a significant role in Chinese teachers' instruction decisions. The Chinese teachers of this study are more likely to address the skills and content which are tested on the high

school entrance exams.

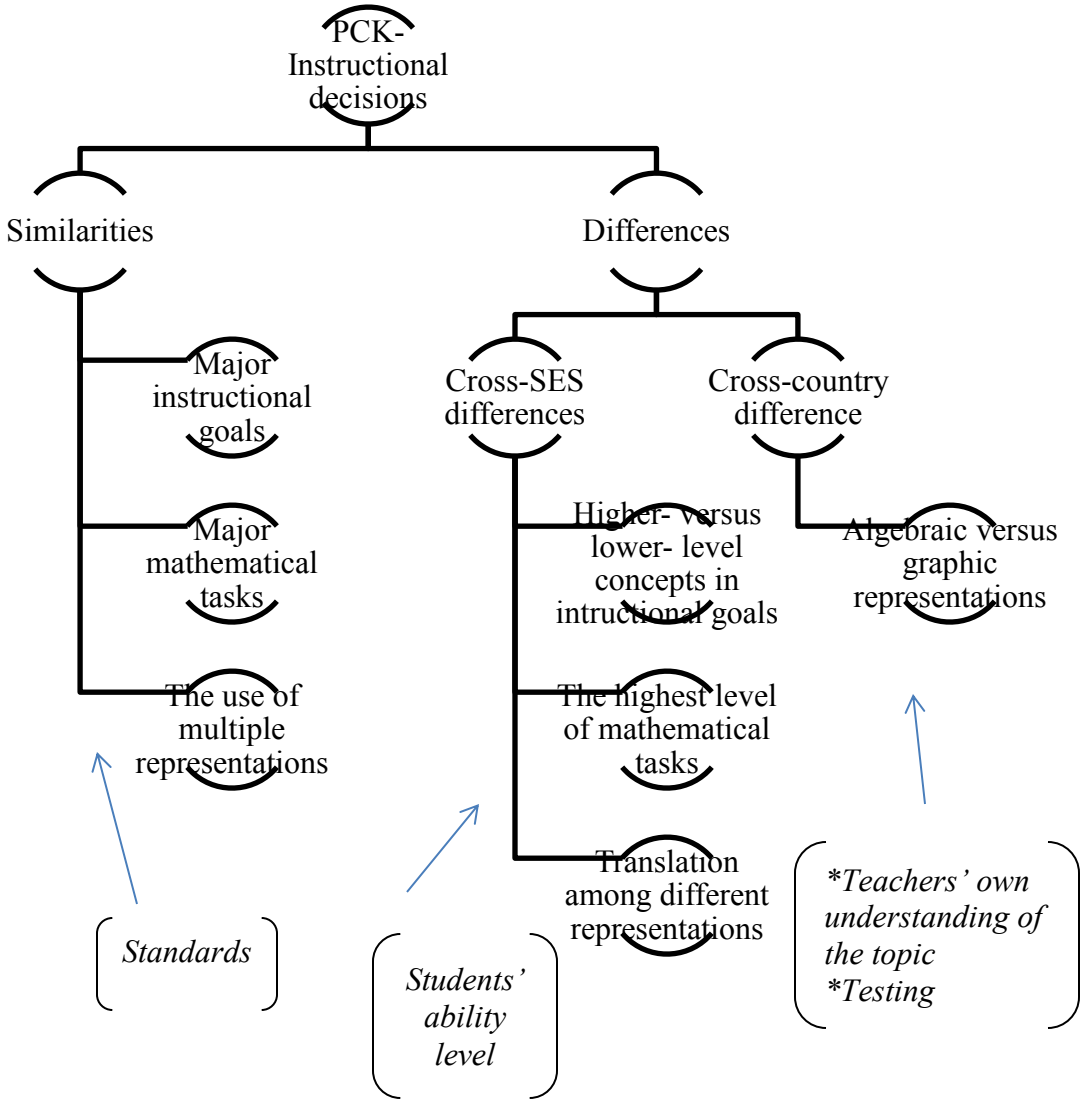


Figure 5 Chart of teachers' instructional decisions

Research Question 2: Dealing with Students' Mistakes

Research question 2 of this study is stated as follows,

How do the U.S. and Chinese higher/lower SES middle school mathematics teachers understand students' mistakes in functions?

Three sub-research questions are put forward to answer this research question.

Mathematical ideas.

The first sub-research question of research question 2 is stated as follows,

What mathematical ideas or concepts do the U.S. and Chinese higher/lower SES middle school math teachers think are important for students to correctly solve the problems in the scenarios?

Table 34

Mathematical Ideas Used in Solving the Problems across Teacher Groups

U.S higher SES	U.S. lower SES	Chinese higher SES	Chinese lower SES
		Scenario 1	
Function definition.(7)	Function definition.(7)	Function definition.(9)	Function definition.(9)
N/A	N/A	Function graphs comes in many shapes.(11)	Function graphs come in many shapes.(6)
Non-linear functions (and graphs).(7)	Non-linear graphs.(4)	Non-linear graphs.(5)	N/A
Linear function features.(5)	Linear function features.(2)	Linear-function features and nonlinear graphs.(6)	Linear function features.(1)
The vertical line test.(5)	The vertical line test.(3)	N/A	N/A
N/A	Input/output tables.(3)	N/A	N/A

N/A	Slop/rise over sun.(3)	N/A	N/A
N/A	Equation determines the shape of a graph.(1)	Equation determines the shape of a graph.(6)	Equation determines the shape of a graph.(3)
		Scenario 2	
The meaning of function graphs in real life.(8)	N/A	The meaning of function graphs in real life.(2)	The meaning of function graphs in real life.(7)
N/A	Meaning of distance, speed and time and relationship between them.(2)	Meaning of distance, speed and time and relationship between them.(8)	Meaning of distance, speed and time and relationship between them.(9)
The meaning of slope/rate of change in graphs.(7)	The meaning of slope/rate of change in graphs.(9)	The meaning of slope/rate of change in graphs.(7)	The meaning of slope/rate of change in graphs.(3)
y-intercept.(4)	y-intercept.(4)	The meaning of y-value on the graph.(4)	N/A
Comparison of numbers and graphs.(1)	Comparisons of numbers and of graphs (2)	Comparison of numbers and graphs (3)	
	Meaning of x-value on the graph.(2)	Meaning of x-value on the graph.(1)	Meaning of x-value and y-value on the graph.(2)
N/A	N/A	Definition of	Definition of

		function.(3)	function.(2)
N/A	N/A	Formula of speed as V=D/T or formula of distance as D=VT.(5)	Equation of speed as V=D/T.(4)
N/A	N/A	Familiar with translation between algebraic functions and graphs.(4)	Familiar with translation between equations and graphs.(1)

Note. Numbers in parentheses indicate the number of responses that fall into that category.

Cross-case results.

As shown in Table 33, in the first scenario in which a student made a mistake in drawing only a line when asked to connect two points on a coordinate plane, the most frequently mentioned mathematical ideas in the four teacher groups are as follows.

In the U.S. higher SES middle school math teachers group, three mathematical ideas were frequently mentioned according to teachers' questionnaire responses. These were first, the definition of function as a rule that assigns to each input exactly one output (see e.g., Questionnaire U.S.-H-#1, #2, & #4); second, non-linear functions (and graphs) (see e.g., Questionnaire U.S.-H-#2 & #6) and linear function features (see e.g., Questionnaire U.S.-H-#2 & #7); and third, the vertical line test in identify function graphs (see e.g., Questionnaire U.S.-H-#5). In the U.S. lower SES middle school teachers group, there were three frequently mentioned mathematical ideas in teachers' questionnaire responses. These were first, function definition that one input had exactly one output (see e.g., Questionnaire U.S.-L-#3); second, the vertical line test in identify functions (see e.g., Questionnaire U.S.-L-#3); and third, non-linear functions graphs (see e.g., Questionnaire

U.S.-L-#2 & #5). In the Chinese higher SES middle school math teachers group, there were four frequently mentioned mathematical ideas which teachers thought were important for correctly solving the problem. These were first, the fact that function graphs come in many shapes (see e.g., Questionnaire Chinese-H-#7); second, linear-function features and non-linear functions and graphs (see Questionnaire Chinese-H-#5); third, function definition as one input having exactly one output (see e.g., Questionnaire Chinese-H-#3); and fourth, the fact that an equation determines the shape of a graph (see e.g., Questionnaire Chinese-H-#7). In the Chinese lower SES middle school math teachers group, three mathematical ideas were frequently mentioned according to these teachers' questionnaire responses. These were first, the fact that function graphs come in many shapes (see e.g., Questionnaire Chinese-L# 9); second, non-linear graphs (see e.g., Questionnaire Chinese-L-#9); and third, function definition as every x with exactly one y in a changing process (see e.g., Questionnaire Chinese-L-#6).

In the second scenario in which a student made a mistake in comparing two lines on a coordinate plane, the most frequently mentioned mathematical ideas that teachers thought were important for correctly solving the problem are as follows.

In the U.S. higher SES middle school math teachers group, there were three frequently mentioned mathematical ideas according to teachers' questionnaire responses. These were first, the meaning of function graphs in real life (see e.g., Questionnaire U.S.-H-#3); second, the meaning of slope or rate of change in real-life situations and in graphs (see e.g., Questionnaire U.S.-H-#6); and third, the meaning of y-intercept (see e.g., Questionnaire U.S.-H-#2). In the U.S. lower SES middle school teachers group, three mathematical ideas were frequently mentioned according to teachers' questionnaire

responses. These were first, the meaning of slope (or rate of change) in graphs (see e.g., Questionnaire U.S.-L-#1); second, the calculation and meaning of y-intercept (see e.g., Questionnaire U.S.-L-#3); and third, the meaning of distance, speed and time and their relationship (see e.g., Questionnaire U.S.-L-#6). In the Chinese higher SES middle school math teachers group, four mathematical ideas were frequently mentioned based on teachers' questionnaire data. These were first, the formula of speed as V=D/T or formula of distance as D=VT (see e.g., Questionnaire Chinese-H-#1); second, the meaning of slope or rate of change in graphs and in the real world (see e.g., Questionnaire Chinese-H-#5); third, the meaning of distance, speed, and time and the relationship between them (see e.g., Questionnaire Chinese-H-#1); and fourth, translation between equations (or algebraic form) and graphs of functions (see e.g., Questionnaire Chinese-H-#7). In the Chinese lower SES middle school math teachers group, three mathematical ideas were frequently mentioned according to these teachers' questionnaire response. These were first, the meaning of function graphs in real life (see e.g., Questionnaire Chinese-L-#1); second, the meaning of distance, speed, and time and their relationship (see e.g., Questionnaire Chinese-L-#5); and third, the equation of speed as V=D/T (see e.g., Questionnaire Chinese-L-#6).

Discussion.

In the first scenario, all the four teacher groups, i.e. U.S. higher SES, U.S. lower SES, Chinese higher SES, and Chinese lower SES middle school math teacher groups, emphasize the importance of understanding the definition of function as "one input with exactly one output" or "for every x there is a unique y", and of knowing non-linear functions and their graphs, in order to correctly solve the problem. These emphases

reflect these teachers' own understanding of the math problem and of function itself, consistent with Ball's argument that teachers' own subject-specific content knowledge is a determinant of their teaching (see Ball et al., 2008). Additionally, when teachers describe function definition as "one input with exactly one output" or "a correspondence between x and y and for every *x* there is a unique *y*" they actually conceive a function from a structural perspective (Sfard, 1991, 1992) or from a correspondence perspective (Schwingendorf et al., 1992), as discussed in the last section.

The Chinese (both higher and lower SES) middle school math teacher groups in this study emphasize the importance of equations or algebraic functions for students to correctly solve the math problem, saying that "equation determines the shape of a graph", whereas almost no U.S. higher SES or U.S. lower SES middle school math teachers mentions the importance of equations or algebraic functions in solving this problem. Interestingly, while the U.S. (both higher and lower SES) middle school math teacher groups in this study emphasize the importance of knowing the vertical line test for correctly solving the problem, no Chinese higher SES or lower SES middle school math teachers mentioned the vertical line test in solving this problem. These cross-country differences indicate different attitudes toward using function representations in teaching by these U.S. and Chinese teachers. While the Chinese teachers in this study pay more attention to equations or the symbolic representation of functions, the U.S. teachers in this study pay more attention to representation of functions in problem-solving, which is basically consistent with Huang and Cai's (2011) argument about Chinese and U.S. teachers' construction of pedagogical representations to teach linear relations. In addition, similar to their finding (Huang & Cai, 2011), both the symbolic and graphic

representations of U.S. and Chinese teachers emphasized in this study might be helpful in solving problems that require students to understand the "function family".

In the second scenario, teachers in all the four teacher groups explicitly state the importance of understanding slope or rate of change in solving this problem. As these teachers explain in their interviews, comparing speed of two objects, both of which are moving at a constant speed, actually involves comparing the slope or rate of change of the two lines, which is the key of the problem. Teachers who point out this idea show their understanding that students need to read the graph from a "global perspective" (Even, 1998) or an "Across-Time perspective" (Monk, 1992), which means students need to be able to understand the patterns of corresponding change between the input variable (time) and the output variable (distance) or to be able to conceptualize functional relationships and make qualitative interpretations of graphs (Dugdale, 1993).

Moreover, all the four teacher groups point out the importance of the connecting graphs of functions to real life situations. Specifically, teachers in the four groups state in their questionnaire responses that it is important to "understand the meaning of functions graphs in the real world" or "understand the meaning of distance, speed and time and their relationship". Based on teachers' explanations, understanding the relationship between distance, time and speed is crucial to understanding the graph. These teachers of the present study have recognized the increasing importance of understanding how (linear) functions can describe real-world (uniform) motions, and of understanding the connections between classroom functions and real-life situations. Functions provide students an opportunity to see how mathematics can describe real-world phenomena (Cooney et al., 1996). From this perspective, teachers in all the four groups in the present

study do realize that in order to correctly read and understand a graph, students need to understand what the graphs represent in a real world. Consistent with Sajka's (2003) argument, teachers in this study are able to recognize the difficulty for students to make this connection since they are rarely given an opportunity to recognize that functions are actually used in the media and in everyday conversations.

Interestingly, both the Chinese higher and lower SES middle school math teacher groups explicitly state the importance of remembering the formula of speed as $V=D/T$ while no U.S. higher or U.S. lower SES middle school math teachers emphasized this. On the other hand, both U.S. higher and lower SES middle school math teacher groups explicitly point out the importance of understanding the *y*-intercept mean on the graph while neither Chinese higher nor lower SES middle school math teachers mentioned that. As discussed above in this section, while the Chinese teachers in this study place more emphases on equations or the symbolic representation of functions, the U.S. teachers in this study hold a higher value for graphic representation of functions in this problem, which is congruent to Cai's (2004) findings on Chinese and U.S. teachers' scoring of students' strategies in solving math problems. The use of symbolic or graphic representations indicates teachers' recognition of the importance of "thinking a problem globally." (McDermott, 1987; Monk, 1992) However, teachers' differential values placed on symbolic representation and on graphic representation indicate their different attitudes toward the use of representations in problem solving.

Summary.

A review of the sub-research question: What mathematical ideas or concepts do the U.S. and Chinese higher/lower SES middle school math teachers think are important for students to correctly solve the problems in the scenarios?

Table 35

Summary for "Mathematical Ideas" Suggested by Teachers to Correctly Solve the Problems

Similarity	Scenario 1: Teachers in all the four teacher groups of this study share in common regarding pointing out the key to solving the problem -definition of function and nonlinear function graphs. Scenario 2: Teachers in all the four teacher groups of this study share in common regarding pointing out the key to solving the problem – meaning of speed, time and distance and their representations on a graph.	
Difference	U.S higher and lower SES: Teachers in the U.S. of this study emphasize the importance of knowing vertical line test for correctly solving the problem.	Chinese higher and lower SES: Teachers in China of this study tend to emphasize the importance of equations or algebraic functions for students to correctly solve the math problem.
	U.S. higher and lower SES: Teachers in the U.S. of this did not mention the formula of speed as V=DT	Chinese higher and lower SES: Teachers in China of this study tend to place more importance on using or remembering the formula of speed as V=D/T or D=VT.

As shown in Table 35, teachers in all the four groups of this study share some commonalities in their suggestions of mathematical ideas or concepts which are used to correctly solve the problems in the two scenarios. As discussed above, in the first

scenario, understanding the definition of function and nonlinear function graphs is regarded as the key to solving the problem in all the four teacher groups; and in the second scenario, understanding the meaning of distance, speed and time and their representations on a graph is regarded as the key to solving the problem in all the four teacher groups. As Ball (see Ball et al., 2008) argued that teachers' subject-specific content knowledge is determinant to their teaching, this finding may reflect these teachers in this study, to a large degree, share in common with respect to their own understanding of the math problems and functions.

The U.S. and Chinese teachers of this study show cross-country differences regarding their emphases on different representations in solving the same problems. As discussed above, while the U.S. teachers in this study hold a higher value for graphic representation of functions (graph and the vertical line test), the Chinese teachers in this study place more emphases on the symbolic representation of functions (algebraic functions and formula) in the two problems, consistent to Huang and Cai's (2011) finding that Chinese teachers tend to emphasize the symbolic representations whereas the U.S. teachers are more likely to value graphic representation to teach linear relations.

Student thinking.

The second sub-research question of research question 2 is stated as follows,

What student thinking do the U.S. and Chinese higher/lower SES middle school math teachers suggest that may lead to the mistakes in the scenarios? Why do these teachers suggest this student thinking?

Table 36

"Students Thinking" Teachers Suggested across Teacher Groups

U.S. higher SES	U.S. lower SES	Chinese higher SES	Chinese lower SES
		Scenario1	
Connecting two points create a line.(5)	Connecting two points create a line.(6)	Connecting two points create a line.(12)	Connecting two points create a line.(8)
A function is a line.(6)	A function is a line (a linear function).(4)	A function is a line (a linear function).(2)	A function is a line.(3)
N/A	Do not understand the problem.(2)	N/A	Do not understand the problem.(3)
Explanations: Students had no experience with non-linear functions; and they did not really understand what a function is (see Interview U.S.-H-#1 & #2).	Explanations: Students had no experience with non-linear functions; and they did not really understand what a function is (see Interview U.S.-L-#2 & #3).	Explanations: Students' fixed-thinking after learning linear functions (see Interview Chinese-H-#1 & #2).	Explanations: Students' fixed-thinking after learning linear functions (see Interview Chinese-H-#1); and they did not have enough experience with non-linear functions (see Interview Chinese-H-#3 & #4).
		Scenario 2	
B is higher at t=2 on the graph	B is higher at t=2 on the graph so B	B is higher at t=2 on the graph- B moves	B is higher at t =2 on the graph so B

which means B moves faster.(6)	moves faster.(2)	faster (4)	movers faster than A (2)
B is higher on the graph (2)	B is higher on the graph (6)	B is higher on the graph.(4)	B is higher on the graph. (9)
B is higher at the starting point.(2)	B is higher at the starting point.(3)	B is higher at the starting point.(3)	B is higher at the starting point.(3)
The points on the graph are the speed.(1)	The speed is represented by the points on a graph.(1)	The points on the graph are the speed.(1)	N/A
N/A	Do not know what speed means.(2)	N/A	N/A
Explanations: Students made Careless mistakes; and comparison of lines (and points) on a graph is a higher-level of math skill (see Interview U.S.-H-#1 & #2).	Explanations: Students' lack of skills in comparison on a graph; and they did not know what the graph really means (see Interview U.S.-L-#3 & #4).	Explanations: Students' carelessness (see Interview Chinese-H-#1, #2, & #4).	Explanations: Students did not know how to read a graph on a coordinate plane (see Interview Chinese-L-#1 & #4); they did not know what x- and y- axis mean (see Interview Chinese-L-#2); and they made careless mistakes (see Interview Chinese-L-#5).

Note. Numbers in parentheses indicate the number of responses that fall into that category.

Cross-case results.

In the first scenario, as shown in Table 36, teachers in the U.S. higher SES middle school math teacher group described two student misconceptions when asked what student might be thinking according to their questionnaire responses. One is "connecting two points create a line and that is function" (see e.g., Questionnaire U.S.-H -#2, #3, & #6); and the other is "a function has to be a line" (see e.g., Questionnaire U.S.-H- #5 & #7). Teachers' reasons for students having these misconceptions were twofold based on the interview data. First, students may lack experience of working with non-linear functions (see Interview U.S.-H-#1 & #2); and second, students may not understand what a function is (see Interview U.S.-H-#1 & #2).

In the U.S. lower SES middle school math teachers group, the most frequently mentioned student thinking or misconception found in teachers' questionnaire responses was "connecting two points create a line" (see e.g., Questionnaire U.S.-L -#1, #2, & #4). Another frequently mentioned student thinking was "a function has to be a line" (see e.g., Questionnaire U.S.-L- #3, #5, & #11). Additionally, two teachers mentioned in their questionnaire responses that their students might "not really understand the problem" (see Questionnaire U.S.-L-#8). Based on teachers' interview data, their explanations of students having these misconceptions pointed to students' lack of experience of working with non-linear functions and of a real understanding of what a function is (see e.g., Interview U.S.-L-#2 & #3).

In the Chinese higher SES middle school math teachers group, the most frequently mentioned student misconception found in teachers' questionnaire responses was "connecting two lines create a line" (see e.g., Questionnaire Chinese-H-#1, #3, #5, &

#7). Additionally, two teachers mentioned "[students might be thinking] a function is a line" (see Questionnaire Chinese-H-#8 & #11) in their questionnaire responses. Based on the interview data, teachers in this group pointed to the "fixed-thinking after learning linear functions," (Interview Chinese-H-#1 & #2), as the major explanation behind this student misconception.

In the Chinese lower SES middle school math teachers group, the most frequently mentioned student thinking or student misconception was "connecting two points make a line and that is a function." (See e.g., Questionnaire Chinese-L-#1, #5, & #7) The other two responses about student thinking were "a function needs to be a line" (see Questionnaire Chinese-L-#2) and "not understand the questions that this problem asked." (See Questionnaire Chinese-L-#4) Based on the interview data, teachers suggested that students had misconceptions mainly because on the one hand, they developed the fixed-thinking after learning linear functions for a long time (see Interview Chinese-H-#1); and on the other hand, students may lack enough experience of dealing with non-linear functions and graphs (see Interview Chinese-H-#3 & #4).

In the second scenario, in the U.S. higher SES middle school math teachers group, the most frequently mentioned student thinking or misconception found in teachers' questionnaire response was "B is higher at t=2 on the graph so B moves faster." (see Questionnaire U.S.-H -#1, #3, #4, & #8) The other two student misconceptions mentioned in this group were "B is higher on the graph" (see e.g., Questionnaire U.S.-H-#2, #5, & #6) and "B is higher at the starting point (see e.g., Questionnaire U.S.-H-#7 & #9). According to the interviews, teachers' explanations for students having this thinking were twofold: students were very likely to make careless mistakes (see e.g., Interview

U.S.-H-#1 & #2), on the one hand; and students may lack experience of comparing lines (and points) on a coordinate plane (see e.g., Interview U.S.-H-#1 & #2).

In the U.S. lower SES middle school math teachers group, the most frequently mentioned student thinking or misconception found in teachers' questionnaire responses was "B looks higher on the graph so B moves faster" (see e.g., Questionnaire U.S.-L- #1, #2, & #4). Another two student thinking or student misconceptions found in their questionnaire responses were "B is higher at t=2 on the graph which means B moves faster," (see e.g., Questionnaire U.S.-L- #3 & #8) and "B is higher at the starting point." (See Questionnaire U.S.-L-#6) According to teachers' interviews, teachers' reasons for students having this thinking were twofold: students may lack skills in comparisons on a graph, on the one hand (see e.g., Interview U.S.-L-#3); and students may not know what the graph really means or represents (see e.g., Interview U.S.-L-#4).

In the Chinese higher SES middle school math teachers group, the most frequently mentioned student thinking or student misconception was "B is higher at t=2 on the graph so B moves faster." (e.g., Questionnaire Chinese-H- #1 & #2) Additionally, another two student misconceptions were mentioned in some teachers' questionnaire responses. These were "B is higher on the graph," (see Questionnaire Chinese-H- #3, #5, & #7) and "B is higher at the starting point." (See Questionnaire-H-#4, #7, & #9) According to teachers' interviews, they suggested that their students were more likely to make careless errors rather than not understand how to solve the problem (see Interview Chinese-H-#1, #2, & #4).

In the Chinese lower SES middle school math teachers group, based on the questionnaire responses, the most frequently mentioned student thinking or student

misconception was “B is higher on the graph” (see Questionnaire Chinese-L-#1 & #2). Another two student misconceptions found in their questionnaire responses were “B is higher at t=2 on the graph so B moves faster,” (see e.g., Questionnaire Chinese-L-#8 & #9) and “B is higher at the start.” (See Questionnaire-H-#3, #4, & #7) According to teachers’ interviews, teachers’ explanations of this student thinking were threefold: first, students may not know how to read a graph on a coordinate plane (see Interview Chinese-L-#1 & #4); second, students may not know what x- and y- axis mean (see Interview Chinese-L-#2); and third, they might be careless when solving this problem (see Interview Chinese-L-#5).

Discussion.

In the first scenario in which a student makes a mistake of drawing only one line between two given points on a coordinate plane, most teachers in each of the four teacher groups think that the student might have a misconception of “connecting two points create a line and that is a function”. In addition, almost half of the teachers in the two U.S. teacher groups and a few teachers in the two Chinese teacher groups come up with student thinking of “a function has to be a line”. These two student misconceptions are most frequently mentioned across all the four teacher groups, indicating teachers in this study have a similar thinking of student thinking across countries and SES. Teachers’ recognition of these student misconceptions of functions in this study indicates teachers’ recognition of the discrepancies between students’ concept image and function definition they are taught (Tall & Vinner, 1981; Vinner & Dreyfus, 1989). Teachers’ explanations, however, about how students obtain these misconceptions are different across countries. The reasons why students obtain these particular misconceptions of functions are the

same according to the U.S. higher and lower SES middle school math teachers' explanations in this study. The most frequently mentioned reason is students' lack of experience of working with non-linear functions. This is partly due to the fact that 7th or 8th graders are not required to understand non-linear functions by the Standards (Common Core) and that teachers do not expect their students to be familiar with non-linear functions. The Chinese higher and lower SES teachers in this study, however, provide a different explanation of their students getting the misconceptions. They expect their students to be familiar with non-linear relations such as inverse-proportional and quadratic functions. They recognize the "fixed-thinking after learning linear functions" as the major reason that leads to student misconceptions. Teachers' understanding of student thinking are likely due to differential beliefs of teachers in two nations (Cai, 2004), which means the different explanations provided by U.S and Chinese teachers in this study may reflect teachers' differential beliefs in students' psychological development in learning math.

In addition, the U.S. higher and lower SES and Chinese lower SES middle school math teacher groups show their concern in their explanations that student may be lack of real understanding of what a function is. According to the explanations by teachers from these three teacher groups, this concern probably derives from the fact that the concept of function itself is extraordinarily complex, on the one hand; and students' ability of understanding a concept at their age, on the other. Differently, teachers interviewed from the Chinese higher SES middle school teacher group do not show this concern. Instead, they show more confidence in their students' ability of understanding the concept of function since "kids are smart". From this perspective, students' ability level and

psychological development at their age are important to teachers' understanding of student thinking (An et al., 2004; Shulman, 1987).

In the second scenario in which a student makes a mistake of taking Object B as the faster one compared with Object A on the position vs. time graphs, most teachers in each of the four teacher groups think that the student might have a conjecture of "B is higher at t=2 on the graph so B moves faster" or "B looks higher on the graph so B moves faster". Teachers' recognition of students' miscellaneous rules in interpreting the graphs indicates that teachers in all four groups of this study show an understanding of students' inclination to be over-literal in interpreting the visual information given in graphs (Bell & Janvier, 1981; Monk, 1992). This "over-literal in interpreting the visual information" can be from a pointwise perspective which emphasizes a specific value of the input/output variable as teachers in the Chinese higher and lower and U.S. higher SES teacher group mention; or can be from a global perspective which emphasizes the overall pattern of change as U.S. lower SES teacher group mentions in this study. The common feature in this study is expecting a "too-close resemblance" between the "prominent visual aspects" (Monk, 1992, p. 176) of the graphs and the real motions that the graphs refer to. Interestingly, teachers' explanations for why students might have this thinking show an across-SES difference rather than show a cross-country difference. Both the U.S. and Chinese higher SES middle school math teacher groups regard carelessness as the major reason, whereas both of the U.S. and Chinese lower SES middle school math teacher groups take "do not know how to interpret graphs" as the major reason for student misconceptions. As explained by the U.S. and Chinese higher SES middle school math teachers in this study, they have beliefs in their students that they understand the

relationship represented in the graphs and are able to correctly solve the problem since their kids are "above the level". Differently, as explained by the U.S. and Chinese lower SES middle school math teachers in this study, their main concern is focused on students' ability of interpreting graphs (and the coordinate plane as well). The difference between teachers' own explanations for students' misconceptions across SES reflects, to a certain degree, teachers' different beliefs in their students' ability level. Obviously, Both U.S. and Chinese higher SES middle school math teachers in this study have higher confidence in their students' ability level whereas U.S. and Chinese lower SES middle school math teachers in this study have lower confidence in their students' ability level. This finding, again, supports the idea that students' ability level is key to teachers' understanding of student thinking, which is similar to the findings in An and her colleagues' (An et al., 2004) and Shulman's (1987) research.

Summary.

A review of the sub-research question: What student thinking do the U.S. and Chinese higher/lower SES middle school math teachers suggest that may lead to the mistakes in the scenarios? Why do these teachers suggest this student thinking?

Table 37

Summary for Similarities/Differences in "Student Thinking" Suggested by Teachers

Similarity	Scenario 1: Teachers in all the four teacher groups suggest similar student misconceptions: "Connecting two points create a line" and "a function has to be a line."
	Differences in Underlying Reasons:
	U.S. teachers suggest students' lack of experience with working with non-linear functions, whereas Chinese teachers suggest students' fixed-thinking after learning linear functions. It may reflect teachers' differential beliefs and expectations in two nations.
	Scenario 2: Teachers in all four groups show an understanding of students' inclination to be over-literal in interpreting the visual information: "B is higher at t=2 on the graph so B moves faster," and "B looks higher on the graph so B moves faster."
	Differences in Underlying Reasons:
	Teachers from higher SES middle schools point to students' carelessness, whereas teachers from lower SES middle schools point to students' lack of ability of correctly interpreting graphs. It may reflect teachers' concerns of students' different ability levels.

As shown in Table 37, teachers in the four groups of this study show similarities regarding their suggestions of student misconceptions in solving the two problems. In the first scenario, "connecting two points create a line" and "a function has to be a line" are two major student thinking or misconceptions appear in teachers' questionnaire responses across all the four groups; and in the second scenario, students' inclination to be over-literal in interpreting the visual information--"B is higher at t=2 on the graph so B moves faster" and "B looks higher on the graph so moves faster"--are two major student

misconceptions across all the four teacher groups. The explanations for these student misconceptions, however, are different.

A cross-country difference is found in the explanations for student misconceptions in the first scenario. While the U.S. teachers of this study tend to suggest that students lack experience of working with non-linear functions, the Chinese teachers tend to suggest that students develop fixed-thinking after learning linear functions. It may reflect teachers' differential beliefs in students' psychological development in learning math across countries.

A cross-SES difference was also found in the explanations for student misconceptions in the second scenario. While teachers from higher SES middle schools of this study point to students' carelessness (or careless errors), teachers from lower SES middle schools point to students' lack of ability to correctly interpret graphs as the major reason. It may reflect teachers' different beliefs in their students' ability level, consistent to the idea that students' ability level is key to teachers' understanding of student thinking (see An et al., 2004; Shulman, 1987).

Correcting strategies.

The third sub-research question of research question 2 is stated as follows,

What approaches would the U.S. and Chinese higher/lower SES middle school math teachers use to correct students' mistakes in the scenarios?

Table 38

Suggested Correcting Strategiese across Teacher Groups

U.S. higher SES	U.S. lower SES	Chinese higher SES	Chinese lower SES
Scenario 1			

Let students draw a couple of graphs to identify functions using definition of function.(3)	N/A	Let students take the lead to discuss which graphs are functions using the definition of function.(3)	N/A
Let students draw a couple of graphs and discuss which are functions using the vertical line test.(2)	Let students draw a couple of graphs and discuss which are functions using the vertical line test.(2)	N/A	N/A
Expose students to a variety of functions in different representations they have seen.(2)	Expose students to a variety of functions in different representations they have seen.(1)	N/A	N/A
Directly use counter-example.(3)	Directly use counter-example.(5)	Directly give students parabolas as counter-examples.(3)	Directly use non-linear graphs.(6)
Make up stories for linear and nonlinear functions and graph them.(1)	Make up stories.(2)	Make up stories/construct scenarios for function definition and graphs.(5)	Make up stories.(1)
N/A	N/A	Use equation of non-linear functions the then graph them.(3)	Use equation of non-linear functions the then

			graph them.(2)
N/A	Reteach the definition of function.(2)	N/A	Reteach the definition of function and its three representations.(5)
		Scenario 2	
Carefully read the graph again to understand the information given in the graph.(4)	Carefully read the graph again to understand the information given in the graph.(1)	Carefully read the graph again to understand the information given in the graph.(6)	Carefully read the graph again to understand the information given in the graph.(1)
Ask students to discuss the concept of slope/rate of change.(3)	Ask students discuss the concept of slope/rate of change.(4)	N/A	N/A
Initiate discussion on steepness and height of lines and how to determine them.(2)	Teach the relationship between speed, time, and distance and initiate a discussion on steepness (slope) and height of lines (starting point) and how to determine these.(5)	Discuss the functional relationship between distance, speed and time on the graphs.(3)	Teach and discuss the functional relationship between distance, speed and time.(6)
Provide real-life examples to	Provide real-life examples to	Provide real-life examples to	Provide real-life examples to

understand the relationship between distance, time and speed.(1)	understand the relationship between distance, time and speed.(2)	understand the relationship between distance, time and speed.(4)	understand the relationship between distance, time and speed.(4)
N/A	N/A	Recall the formula of speed and calculate the speed.(1)	Ask students to remember the formula of speed.(3)
Use a table of values to help compare.(1)	N/A	N/A	N/A

Note. Number in THE parentheses indicates the number of responses that fall into that category.

Cross-case results.

As shown in Table 38, in the first scenario, most teachers in the U.S. higher SES middle school math teacher group chose one of the following two approaches to correcting their students' misconception(s). One is providing non-linear function graph examples (see e.g., Questionnaire U.S.-H-#8); and the other is letting students draw a couple of graphs and discussing with class which are functions using the vertical line test (see e.g., Questionnaire U.S.-H-#6). In the U.S. lower SES middle school math teachers group, most teachers used the approach of providing examples of non-linear function graphs (see e.g., Questionnaire U.S.-L-#1) to help students correct the mistake. The others were primarily letting students draw a couple of graphs and discussing which are functions using the vertical line test; and re-teaching the definition of function (see e.g., Questionnaire U.S.-L-#7). In the Chinese higher SES middle school math teachers group,

most teachers chose to make up stories or construct scenarios and graphs to help students understand functions and corresponding graphs (see e.g., Questionnaire Chinese-H-#1). Another three correcting approaches were found to be frequently used based on their questionnaire responses. First, let students take the lead to discuss which graphs are functions using the definition of function (see e.g., Questionnaire Chinese-H-#5); second, use equations of non-linear functions and then graph them (see e.g., Questionnaire Chinese-H-#7); and third, directly give students parabolas a counter-examples (see e.g., Questionnaire Chinese-H-#9, #12). In the Chinese lower SES middle school math teachers group, two approaches are frequently used in correcting students' mistake based on teachers' questionnaire responses. One is directly using counter-examples of function graphs (see e.g., Questionnaire Chinese-L-#1); and the other is re-teaching the definition of function and its three forms of representations of functions (see e.g., Questionnaire Chinese-L-#6).

In the second scenario, most teachers in the U.S. higher SES middle school math teacher group chose one of the following two approaches to correct their students' mistake. One is asking students to carefully read the graph again to understand the information given in the graph (see e.g., Questionnaire U.S.-H-#2 & #3); and the other is leading students to discuss what y-intercept and slope mean (see e.g., Questionnaire U.S.-H-#9). In the U.S. lower SES middle school math teachers group, most teachers chose the following approach to correct their students' mistake: Teaching the relationship between speed, time, and distance and initiating a discussion on steepness (slope) and height of lines (starting point) and how to determine these (see e.g., Questionnaire U.S.-L-#1). Another approach that teachers in this group frequently used is asking students to discuss

the concept of slope or rate of change in class (see e.g., Questionnaire U.S.-L-#4). In the Chinese higher SES middle school math teachers group, the most frequently used approach is asking students to carefully read the graph again to understand the information given in the graph (see e.g., Questionnaire Chinese-H-#1). Another two approaches that are frequently used in correcting students' mistake are: first, providing real-life examples for students to understand the relationship presented in the graph (see e.g., Questionnaire Chinese-H-#7); and second, discussing the functional relationship between distance, time and speed with class (see e.g., Questionnaire Chinese-H-#5). In the Chinese lower SES middle school math teachers group, three correcting approaches were found to be frequently used based on their questionnaire responses: First, teaching and discussing the functional relationship between distance, time and speed (see e.g., Questionnaire Chinese-L-#1); second, providing real-life examples to students to understand the relationship presented in the graph (see e.g., Questionnaire Chinese-L-#5); and third, asking students to remember the formula or equation of speed (see e.g., Questionnaire Chinese-L-#10).

Discussion.

In the first scenario, all the four teacher groups in this study mention the use counter-examples of function graphs or non-linear function graphs to help students correct their mistakes as the most important approach since this approach is straightforward (and easy) for students to understand their mistake according to teachers' own explanations.

Moreover, all the four teacher groups mention the instruction of function definition to help students correct their mistake on graphing functions. It is explicitly

stated in teachers' own explanations in this group that understanding the concept (and definition) of function are crucial to correct students' mistake in this problem. Teachers from all the four teacher groups may be equally likely to elaborate the concept (and the definition) of function when dealing with their mistake on graphing functions. This contradicts Ma's (1999) finding that Chinese teachers are more likely to use the approach of elaborating the concept of place value when dealing with students' mistake on multi-digit number multiplications.

Interestingly, while both the Chinese higher and lower SES middle school math teacher groups emphasize that using equations or algebraic functions and then graphs as an important approach to deal with students' mistake, the U.S. higher SES middle school teacher group emphasizes the use of the vertical line test as important in identifying function graphs when dealing with the same students' mistake. It is congruent to previous research finding that Chinese teacher are more likely to construct algebraic approaches (see Cai & Wang, 2006). According to teachers' own explanations, the reason why the Chinese teachers in this study emphasize algebraic functions in correcting students' mistake lies in their emphasis on the symbolic representation in teaching the topic of functions as well as the role of symbolic representation in tests and in developing students' abstract thinking. The reason why the U.S. higher SES teacher group in this study emphasizes the vertical line test in dealing with students' mistake lies in their emphasis on the representation in teaching functions. Thus, on the one hand, teachers' decision on approaches to correcting students' mistake in this study is, to a certain degree, consistent with what they think is important for teaching and learning the topic of function. On the other hand, the use of equations or symbolic representation of functions

is seen as an important modeling approach in advancing students' abstract thinking for the both the Chinese higher and lower SES middle school teacher groups.

In the second scenario, the most frequently mentioned correcting approach by the teachers in the U.S. and Chinese higher SES middle school math teacher groups is to ask their students to carefully read the graph again to understand the information given in the graph to correct students' mistake in this problem. It is obvious that, based on the explanations provided by the teachers in the U.S. and Chinese higher SES teacher groups, the use of this approach is primarily due to these teachers' understanding of their students (and their students' thinking). That is, as explained by these teachers, their students are at a higher level and the main reason of making this mistake should be the carelessness rather than the lack of understanding of the concepts or the relationship. From this perspective, the U.S. higher SES teacher group' approach to dealing with this mistake has some aspects in common with that of the Chinese higher SES teacher group in this study. They share the same major correcting approach on the one hand; and share the same explanation of constructing this approach which emphasizes students' higher ability level, on the other.

Differently, the most frequently mentioned approach of correcting students' mistake by the teachers in the U.S. and Chinese lower SES middle school math teacher groups in this study is to teach their students the relationship between speed, time, and distance with a discussion on the steepness and height of a line and on how to determine these. According to the explanations provided by the teachers from these two groups, the use of this approach is mainly due to teachers' worries whether their students really understand the meaning of speed, time and distance and their relationship which is central

to correctly solving this problem. As explained by these teachers, their students are at a relatively lower ability level and less likely to understand the relationship presented in the graph. Thus, it is important for these teachers to teach the meaning of speed, time and distance and their relationship as an important approach of correcting this mistake. Similar to what discussed above, the U.S. lower SES teacher group' approach to dealing with this mistake shares some aspects in common with that of the Chinese lower SES teacher group in this study. They construct similarly the major correcting approach on the one hand; and provide the similar explanation of constructing this approach which emphasizes students' lower ability level, on the other. From this perspective, the difference between teachers' correcting approaches across SES in this study can be explained by these teachers' understanding of their students' ability levels. Students, including students' ability levels and students' conceptions and misconceptions, stay at the heart of teachers' understanding of student thinking (e.g., Carpenter et al., 1988; Shulman, 1987). The cross-SES difference in constructing correcting approach and teachers' explanations for this difference in this study supports this importance of students' ability levels in teachers' understanding of students.

Additionally, both the Chinese higher and lower SES middle school math teacher groups mention the use of real-life examples to understand the relationship as an important approach to correct students' mistakes on comparing the two lines presented in the graph. One explanation provided by the Chinese lower SES middle school math teacher group is to use real-life examples to "hook up" students' interest. Another explanation provided by the Chinese higher SES middle school math teacher group is that real-life examples connect functions and real-world contexts as to "make math learning

meaningful". The Chinese teachers' use of real-life examples and their explanations in this study may reflect the current efforts in China that making math learning meaningful for students, which is also stated in the standards that helping students build connection between math learning and real-world contexts (Chinese Department of Education, 2011).

Summary.

A review of the sub-research question: What approaches would the U.S. and Chinese higher/lower SES middle school math teachers use to correct students' mistakes in the scenarios?

Table 39

Summary for Similarities/Differences in "Correcting Strategies" Suggested by Teachers

Similarity	Scenario 1: All the four teacher groups in this study mention the use of counter-examples of function graphs and the use of definition of function to help students correct their mistakes.	
Difference	U.S higher and lower SES groups: Scenario 1: Emphasize the use of the vertical line test as important in identifying function graphs when dealing with the mistake.	Chinese higher and lower SES groups: Scenario 1: Emphasize the use of equations/algebraic functions and then graphs as an important approach to deal with this mistake.
	U.S. and Chinese higher SES groups: Scenario 2: Ask students to carefully read the graph again to understand the information given in the graph to correct students' mistake in this problem.	U.S. and Chinese lower SES groups: Scenario 2: Teach their students the relationship between speed, time, and distance with a discussion on the steepness and height of a line and on how to determine these.

	Underlying reason: Students are at a higher level and the main reason of making this mistake should be the carelessness rather than the lack of understanding of the concepts or the relationship.	Underlying reason: Students are at a relatively lower ability level and less likely to understand the relationship presented in the graph.

As shown in Table 39, teachers in all the four teacher groups of this study share some similarities with respect to the approaches of correcting students' mistake in the first scenario. They all mention the use of nonlinear function graphs and the use of definition of function to help students correct their mistake. Based on teachers' interview data, the use of non-linear function graphs seems to be a straightforward (and easy) approach. The instruction on the definition of function, according to teachers' explanations, addresses the conceptual understanding of functions in correcting students' mistake on graphing functions. Inconsistent with Ma's (1999) finding, the U.S. and Chinese teachers of this study are equally likely to use the approach of elaborating the concept when dealing with students' mistake on graphing functions.

Two differences are found regarding teachers' correcting approaches. One is a cross-country difference which is found in the first scenario. While the U.S. teachers of this study tend to emphasize the use of the vertical line test in correctly identifying function graphs, the Chinese teachers of this study tend to place more values on the role of equations or algebraic functions in determining function graphs when dealing with students' mistake. Teachers' own understanding of the content influence their teaching practice. The difference may reflect teachers understanding of the role of different math languages in math teaching and learning in the two nations.

The other one is a cross-SES difference which is found in the second scenario. According to teachers' questionnaire responses, while teachers from the higher SES middle schools of this study are more likely to ask students to carefully read the graph again to understand the information given in the graph, teachers from the lower SES middle schools of this study are more likely to formally teach their students the relationship between speed, time, and distance and their graphs to help students correct their mistake. Based on teachers' explanations in the interviews, students' ability level is the major reason behind these decisions. Students from higher SES schools are believed to be at a higher ability level and it's highly possible that the carelessness rather than the lack of understanding of the concepts or the relationship leading to their mistake; whereas students from lower SES schools are believed to be at a relatively lower ability level and less likely to understand the relationship presented in the graph.

Concluding remarks for research question 2.

How do the U.S. and Chinese higher/lower SES middle school mathematics teachers understand students' mistakes in functions?

I find that in this study, teachers from both countries and both socio-economic statuses share some similarities in dealing with students' mistakes. First, they tend to point out similar key concepts or ideas for correctly solving the problems in the two scenarios, which may reflect that they share similar understanding of the problems and related math topics. Second, they tend to recognize similar student misconceptions in solving the problems in the two scenarios. They not only recognize the discrepancies between students' concept image and function definition as represented in the first scenario; but also recognize students' inclination to be over-literal in interpreting visual

information in the second scenario. Lastly, they are equally likely to construct straightforward (and easy) approach (i.e. providing non-linear graphs) and emphasize conceptual understanding of functions (i.e. instruction on function definition) to help students correct their mistake in the first scenario.

I find two major cross-SES differences regarding teachers' dealing with students' mistakes primarily in the second scenario. One lies in the reasons that may cause students to make the mistake. While teachers from the higher SES middle schools of this study point to students' carelessness, teachers from the lower SES middle schools point to students' lack of ability to correctly interpret the graphs. The other, as a result, lies in the approaches teachers suggest to correct the mistake. While most of the teachers from the higher SES middle schools suggest that their students only need to carefully read the graph again and they will correctly solve the problem, most of the teachers from the lower SES middle schools of this study suggest a formal instruction on the relationships presented in the linear graph. These differences may reflect teachers' consideration of their students' ability level when dealing with students' misconceptions and mistakes. It can be arguably stated that teachers' concerns on students' average ability levels contribute, to a certain extent, to the differences in teachers' understanding of students' mistakes between those who are from higher SES schools and who are from lower SES schools.

I also find cross-country differences between the U.S middle school math teachers and the Chinese middle school math teachers in this study. One major difference is that when suggesting math concepts and correcting approaches, the U.S teachers of this study are more likely to emphasize the use of graphic functions, whereas the Chinese teachers

of this study tend to emphasize the use of algebraic or symbolic functions (e.g., formula and equations) in solving problems. It may again reflect teachers' own understanding of the content and their beliefs in the use of different math languages in math teaching and learning. The other major difference lies in teachers' explanations for student misconception of function graphs. While the U.S. teachers of this study tend to point to students' lack of experience of working with non-linear functions, the Chinese teachers of this study tend to point to students' fixed-thinking after long-time linear-functions learning. It may reflect teachers' differential beliefs in students' psychological development in learning math across the two nations.

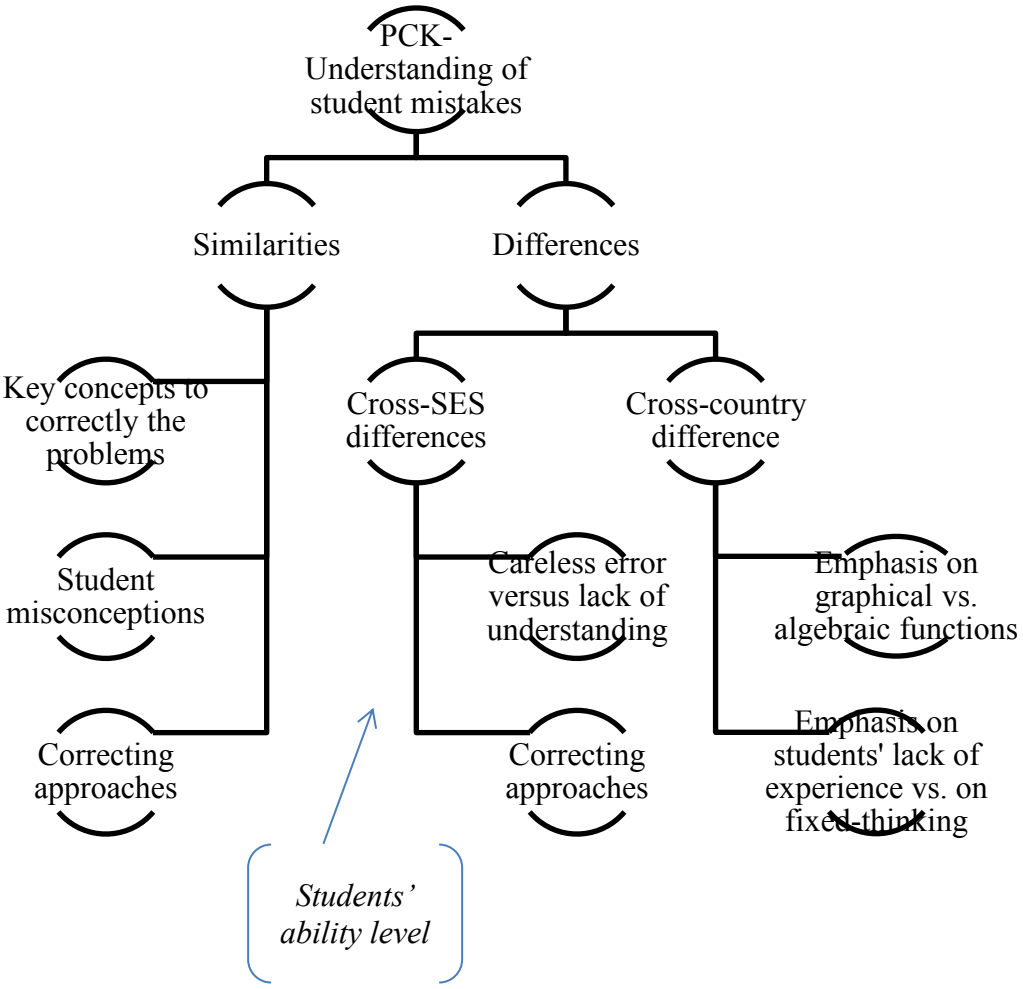


Figure 6 Chart of teachers' understanding of students' mistakes in functions

Research Question 3: Curricular Knowledge

Research question 3 of this study is stated as follows,

What knowledge do the U.S. and Chinese higher/lower SES middle school mathematics teachers have about the mathematics curriculum?

What instruction materials (textbook in particular) do they use and how do they use them?

How do they understand functions as related to other subject areas?

How they understand functions as related to other topics in math learning?

Instructional materials.

The first sub-research question of research question 3 is stated as follows,

What instruction materials (textbook in particular) do these teachers use and how do they use them?

Table 40

Instructional Materials Used by Teachers across Four Groups

U.S higher SES	U.S. lower SES	Chinese higher SES	Chinese lower SES
Standards.	Standards.	Standards.	Standards.
Textbooks*: Glenco Algebra*; *Holt*; *McDougal Littell Algebra*; *NCTM illuminations*.	Textbooks: *Connected Mathematics Project*; *Glencoe Algebra*; *McDougal Littell Algebra*; *Holt.*	Textbooks: *Renjiaoban Mathematics*; *Beijing Kegaiban Mathematics*.	Textbooks: *Renjiaoban Mathematics*; *Huashidaban Mathematics*; and *Beijing Kegai Mathematics*.
Teacher created materials.		Outside materials.	
Outside materials	Teacher created materials.		Outside materials.
	Outside materials.		

Cross-case results.

As shown in Table 40, based on their questionnaire responses, there are four types of instructional materials used by the U.S. higher SES middle school math teacher group. These were the standards; textbooks, such as *Glencoe Algebra*, *Holt* and *McDougal Littell Algebra*; teacher-created materials; and outside materials. Similarly, there were

four types of instructional materials used by the U.S. lower SES middle school math teacher group. These were the standards; textbooks, such as *Connected Mathematics Project*, *Glencoe Algebra* and *McDougal Littell Algebra*; teacher-created materials; and outside materials. In the Chinese higher SES middle school math teacher group, three types of instructional materials were found to be used in the classrooms. These were the standards; textbooks, mainly *Renjiaoban Mathematics* (and *Beijing Kegaiban Mathematics*); and outside materials. Similarly, in the Chinese lower SES middle school math teacher group, three types of instructional materials were found. These were the standards; textbooks, mainly *Renjiaoban Mathematics* (and *Huashidaban Mathematics* and *Beijing Kegai Mathematics*); and outside materials.

In the interviews, when asked about how they used their textbooks in class, teachers in both the U.S. higher and lower SES middle school math teacher groups explained that they used only a few examples or problems from their designated textbooks. The reasons behind how the U.S. higher SES middle school math teachers used their textbooks were twofold: outdated examples or problems in the textbooks, as Ms. Baker and Mr. Carter explained (see Interview U.S.-H-#1 & #2); and AIMS test pressure, as Ms. Baker explained (see Interview U.S.-H-#1). The major reason for not following exactly the textbooks, provided by the U.S. lower SES middle school math teacher group, was that their textbooks did not present the topics in an acceptable way that these teachers may feel comfortable with, as Mr. Jordon and Ms. Haley explained in their interviews (see Interview U.S.-L-#3 & #5). Teachers in the Chinese higher and lower SES middle school math teacher groups explained that they used some (or basic) examples or problems from their textbooks and that they use supplementary problems

from their outside materials. The reasons for their use of the textbooks provided by the Chinese higher SES middle school teacher group included first, teachers felt dissatisfied with the problems provided in the textbooks (see Interview Chinese-H-#1); and second, the problems provided in the textbooks cannot fully cover the test items on the high school entrance exam (see e.g., Interview Chinese-H-#3 & #4). The reasons provided by the Chinese lower SES middle school math teacher group included the basic level of problems in the textbooks (see e.g., Interview Chinese-L-#2, #3, & #5) and the pressure from the high school entrance exam (see e.g., Interview Chinese-L- #3 & #5).

Discussion.

In this study, overall, the types of instructional materials used by the U.S. and Chinese middle school math teachers for teaching the topic of functions are similar. Standards (i.e. *Arizona Common Core Standards* and the *New Mathematics Standards* in China); designated or assigned textbooks by each school or school district (e.g., *Glencoe Algebra* and *McDougal Littell Algebra* in the U.S. and *Renjiaoban Mathematics* in China); and outside materials are three major types of instructional materials which are frequently used together in class by the U.S. and Chinese teachers in this study. First of all, it is obvious that the math standards are commonly used in teaching. Almost all the teachers, across country and SES in this study, mention that they use standards as guide in class. It may reflect the values that teachers place on the standards in the present study. It may also reflect that the standards-based instruction has become a "norm" in teachers' knowledge in both countries, that is, "one needs to teach based on what is required in the standards".

Second, generally, the U.S. and Chinese middle school math teachers in this study do have textbooks which are designated or assigned by their school or school district. However, based on the variety of textbooks teachers provide in this study, the textbooks the Chinese middle school math teachers used in class are more uniform compared to their counterparts in the U.S. In Beijing where this study is conducted, primarily three textbooks for teaching the middle school math – *Renjiaoban Mathematics*, *Huashidaban Mathematics*, and *Beijing Kegaiban Mathematics* - are used by teachers. Among these three textbooks, *Renjiaoban Mathematics, which is published by* the People's Education Press, is the most widely used one by teachers. In this study, almost all the teachers in the Chinese higher and lower SES middle school teacher groups mention they use the *Renjiaoban Mathematics* as textbook in class. In contrast, the pool of textbooks used by the U.S. higher and lower SES middle school math teachers in the Phoenix Metro area is fairly larger in this study. Textbooks used in the classroom include *Connected Mathematics Project*, *Glencoe Algebra*, *McDougal Littell Algebra*, and *Holt Mathematics*, *NCTM Illuminations*, and so forth. It is also noteworthy that there are some teachers in both the U.S. higher and lower SES middle school math teacher groups mention that they do not have textbooks in classroom; instead, they use their own created materials. This phenomenon is not found in the Chinese higher and lower SES middle school math teacher groups. This may reflect that the U.S. middle school math teachers in this study have more freedom in choosing and using textbooks compared to their Chinese counterparts.

Third, the U.S. and Chinese higher and lower teacher groups in this study share some commonalities in regard to their use of the textbooks. Based on the interviews with

teachers from the four groups, it is obvious that on the one hand, teachers in this study, across country and SES, tend to use a few or some rather than follow all examples or problems in the textbooks; and outside materials are important in complementing the textbooks on the other. Specifically, teachers in both the U.S. higher and lower SES middle school math teacher groups point out that they use only a few examples and problems in the textbooks and use a lot of examples and problems from other materials. Similarly, teachers in both the Chinese higher and lower SES middle school math teacher groups point out that they use basic-level examples and problems from their textbooks and they use higher-level examples and problems from other materials. Why do teachers in this study use their textbooks in this particular way? Explanations for teachers' use of textbooks are summarized as follows. The first explanation lies in these teachers' dissatisfaction with their textbooks. Teachers in the U.S. higher and lower SES teacher groups articulate that some of the examples and problems in the textbooks are outdated, or the ways the textbooks present the problems are not appropriate for their students. This finding is congruent to Li's (2000) argument that the U.S. textbooks presented a relatively outdated curriculum compared to some other countries. It may indicate the necessity of continuing efforts on the reform of curriculum and textbook in particular since textbooks do not reach the expectations of teachers who stand at the frontline of teaching. The second explanation is found from the Chinese higher and lower SES teacher groups. That is, examples and problems presented in the textbooks only cover the basic requirement of the standards. It is necessary to expose students to higher-level problems in class in order to help them develop stronger problem-solving skills and achieve higher on the high school entrance exam. The Chinese teachers in this study are

then motivated to use higher-level problems which are frequently found in other materials such as teacher created materials and outside materials. From this perspective, it can be arguably stated that the test pressure especially the high school entrance exam pressure plays a significant role in teachers' selection and use of instructional materials. This pressure can also be found in the U.S. higher SES middle school teacher group that in order to maintain a high level of AIMS performance, it is necessary to use a lot of materials instead of textbooks only to expose students to problems similar to test items. Although this phenomenon is not found in the U.S. lower SES middle school math teacher group in this study, it may still reflect the increasing influence of high-stakes testing on teachers' instruction in the U.S.

Summary.

Table 41

Summary for Similarities/Differences in the Use of Instructional Materials by Teachers

Similarity	Math standards (i.e. *Arizona Common Core Standards* and the *New Mathematics Standards* in China) are used as guide in all the four teacher groups.	
	Teachers tend to not follow all examples or problems in the textbooks; instead, they use only a few or some examples and supplement problems from outside materials.	
	High-stakes testing pressure – AIMS in Arizona and the high school entrance exam in Beijing- as explaining teachers' use of textbooks.	
Difference	U.S. higher and lower SES:	Chinese higher and lower SES:
	The assigned textbooks in Arizona public schools are more diverse.	The assigned textbooks in Beijing public schools are more uniform.

	U.S. Higher and lower SES: Some of the examples and problems in the textbooks are outdated, or the ways the textbooks present the problems are not appropriate for their students.	Chinese higher and lower SES: Examples and problems presented in the textbooks only cover the basic requirement of the standards and the high school entrance exam.

As shown in Table 41, teachers in this study have the following similarities regarding their use of instructional materials. First, math standards are used as guide in the classrooms of all the four teacher groups, which may reflect that the standards-based instruction has become a "norm" in teachers' knowledge in both the U.S. and China. Second, teachers tend to not follow all examples or problems in the textbooks. Instead, they use only a few or some examples and supplement problems from outside materials. Third, teachers of this study share in common with respect to their recognition of the increasingly important part that high-stakes testing (AIMS in Arizona and high school entrance exam in Beijing) played in their use of textbooks.

One cross-country difference is found in the variety of designated textbooks used in the schools in Arizona and Beijing. The textbooks the Chinese middle school math teachers used in class are more uniform compared to their counterparts in the U.S. This may reflect that the U.S. middle school math teachers in this study have more freedom in choosing and using textbooks compared to their Chinese counterparts. The other cross-country difference is found in teachers' explanations of why they do not fully follow the example in the textbooks. The U.S. teachers of this study suggest that the examples and problems in the textbooks are outdated, which is not found in the Chinese teachers' explanations. This is congruent to Li's (2000) argument that the U.S. textbooks presented

a relatively outdated curriculum compared to some other countries. The Chinese teachers of this study suggest that the examples and problems presented in the textbooks only cover the basic requirement of the standards and the high school entrance exam, which may again imply the significant influence of testing on teachers' teaching practice in China.

Lateral and vertical curriculum knowledge.

The second and third sub-research questions of research question 3 are stated as follows,

How do teachers understand functions as related to other subject areas and how do they understand functions as related to other topics in math learning?

Table 42

Lateral and Vertical Curriculum Knowledge across Four Teacher Groups

U.S. higher SES	U.S. lower SES	Chinese higher SES	Chinese lower SES
Lateral curriculum knowledge			
Math and science are intertwined; interdisciplinary work goes less intense as it used to be.	Little interdisciplinary work; and include physics concepts in class.	Very few cooperation; some simple physics concepts included.	Very little cooperation; and include some simple physics topics in class.
Vertical curriculum knowledge			
Functions a part of the relationship system.	Pattern, relation and function are connected.	The topic of functions as at the highest level of students' math	The topic of functions as the most important one in students'

	learning at school. All the math knowledge is connected.	math learning in school. Math topics are connected.

Cross-case results.

Based on the interview data, in the U.S. higher SES middle school teachers group, when asked about their interdisciplinary work between functions (math) and other disciplines, teachers provided the following explanations. Mr. Carter and Ms. Denison pointed to the fact that math and science were intertwined and the efforts in incorporating science into math learning (see e.g., Interview U.S.-H-#2 & #3). Differently, Ms. Baker pointed out that interdisciplinary work went less intense as it used to be (see e.g., Interview U.S.-H-#4). In the U.S. lower SES middle school teachers group, teachers pointed out that they now did little interdisciplinary work (see e.g., Interview U.S.-L-#5) except for including some physics concepts in class (see Interview U.S.-L-#1). In both the Chinese higher and lower SES middle school math teachers groups, teachers explained that their math teaching was barely related to other classes in school except for incorporating some simple physics concepts in class (see e.g., Interview Chinese-H-#2 & #3; Interview Chinese-L-#2, #3).

According to the interview data again, in both the U.S. higher and lower SES middle school math teachers groups, when asked to describe how they saw the topic of functions being related to other topics in students' math learning in school years, teachers explained that they viewed functions as a way of describing relationships. Specifically, teachers in the U.S. higher SES middle school math teacher group pointed out that

functions are part of the relationship system, as Mr. Carter and Ms. Edson explained (see e.g., Interview U.S.-H-#2 & #4). Teachers in the U.S. higher SES middle school math teacher group pointed out that patterns, relations and functions were closely connected, as Ms. Gerold and Ms. Kean explained in the interviews (see e.g., Interview U.S.-L-#2 & #6). Teachers in both the Chinese higher and lower SES middle school math teachers groups pointed to the importance of the topic of functions in students' math learning from elementary school to high school and the connection between all the math knowledge. Specifically, as explained by Ms. Zhao and Ms. Sun who were from the Chinese higher SES middle school group, the topic of functions was at the highest level of students' math learning at school (four levels: number; expression; set of expressions; and change, correspondence and functions) and all the math knowledge was connected (see e.g., Interview Chinese-H-#1 & #3). In the Chinese lower SES middle school math teachers group, as Mr. Wu and Ms. Zheng explained, the topic of functions was the most important one in students' math learning in school and all math topics were actually connected (see e.g., Interview Chinese-H-#2 & #3).

Discussion.

According to teachers' descriptions, interdisciplinary collaboration is not frequent in the U.S. and Chinese classrooms in this study. Teachers in all the four math teacher groups explicitly state that they do little interdisciplinary work together with teachers from other disciplines except for incorporating some physics concepts into their class. Math and science are still separated in their respective classrooms. The collaboration between math teachers and science teachers (physics teachers in China) is rarely found. The "interdisciplinary work" math teachers usually do is to incorporate some physics

concepts in their math classrooms rather than work with science (physics) teachers. Additionally, In China, physics teachers expect math teachers to teach functions earlier such that they can teach related content in their physics classrooms. This may reflect the atmosphere in China in which math and physics are closely connected, or more accurate, students' performance in physics largely depends on their performance in math.

How do teachers see the topic of functions being related to other topics in students' past and future math learning? Teachers in all the four teacher groups have the awareness of seeing the learning of functions in a connected network. The difference is that while the U.S. middle school math teachers tend to view the topic of functions in a network of patterns, relations and functions, the Chinese middle school math teachers are more likely to view the topic of functions in a network of the entire math knowledge. It may reflect the different manners that the topic of functions is introduced in the U.S. and China (Howson, 1995). It may also reflect that teachers' math knowledge packages vary from the U.S. and China (Ma, 1999).

Summary.

Table 43

Summary for Similarities/Differences in Teachers' Lateral and Vertical Curriculum Knowledge

Similarity	Lateral curriculum knowledge: Interdisciplinary collaboration is not frequent in the U.S. and Chinese classroom in this study. Vertical curriculum knowledge: Teachers in all the four teacher groups of this study have the awareness of seeing the learning of functions in a connected network.	
Difference	The U.S. higher and lower SES:	Chinese higher and lower SES:

	Teachers tend to view the topic of functions in a network of patterns, relations and functions.	Teachers are more likely to view the topic of functions in a network of the entire math knowledge.

As shown in Table 5.16, teachers in all the four groups share in common regarding both their lateral and vertical curriculum knowledge. Interdisciplinary collaboration is still rare found in both the U.S. and Chinese classroom of this study. Only a few physic concepts are introduced to the math classroom. It may reflect that math and science are still separated in their respective classrooms. Moreover, teacher in both the U.S. and China are increasingly aware the importance of seeing the learning of functions in a connected network.

The only difference regarding teachers' vertical curriculum knowledge is that while the U.S. teachers of this study tend to view the topic of functions in a network of patterns, relations and functions, the Chinese teachers are more likely to view the topic of functions in a network of the entire math knowledge. It may reflect that teachers' math knowledge packages vary from the U.S. to China, consistent with Ma's (1999) previous findings.

Concluding remarks for research question 3.

What knowledge do the U.S. and Chinese higher/lower SES middle school mathematics teachers have about the mathematics curriculum regarding functions?

I find in this study that teachers, across countries and socio-economic statuses, share quite a few commonalities regarding their curricular knowledge. First, they tend to use the same type of instructional materials (i.e. standards, textbooks, teachers-created materials and outside materials) in their classroom. Standards guide their teaching

practice but textbooks are not fully used in classrooms. Second, both the U.S. and Chinese teachers of this study have the pressure to address the challenges on the tests through their use of instructional materials (*AIMS* in Arizona and *High School Entrance Exam* in Beijing), indicating their recognition of the increasingly important part that high-stakes testing played in teaching practice. Third, interdisciplinary collaboration is rarely seen in today's classrooms, which indicates a separation of math and other discipline (science in particular) in the U.S. and Chinese classrooms. Fourth, teachers all tend to view the learning of functions (and mathematics) in a connected network. They recognize that math is connected.

I also find cross-country differences, with respect to the diversity of textbooks, reasons behind textbooks use and the coverage of math network related to functions, between the U.S. and Chinese teachers of this study. First, the textbooks in the U.S. classrooms of this study seem to be more diverse compared to their counterparts in China, which may reflect that the U.S. middle school math teachers may have more freedom in choosing textbooks compared to their Chinese counterparts. Second, while the U.S. teachers suggest that the examples and problems in the textbooks are outdated, indicating that the U.S. textbooks present a relatively outdated curriculum compared to some other countries; the Chinese teachers suggest that problems presented in textbooks are too easy to address the challenges in the high school entrance exam, indicating that the Chinese teachers of this study are more likely to be heavily influenced by testing. Lastly, although teachers in the two nations tend to view the topic of function as connected to other topics, how they are connected varies across countries. While the U.S. teachers of this study are more likely to view the topic of functions in a network of patterns, relations and functions,

the Chinese teachers are more likely to view the topic of functions in a network of the entire math knowledge, which may reflect that teachers' math knowledge packages vary from the U.S. to China.

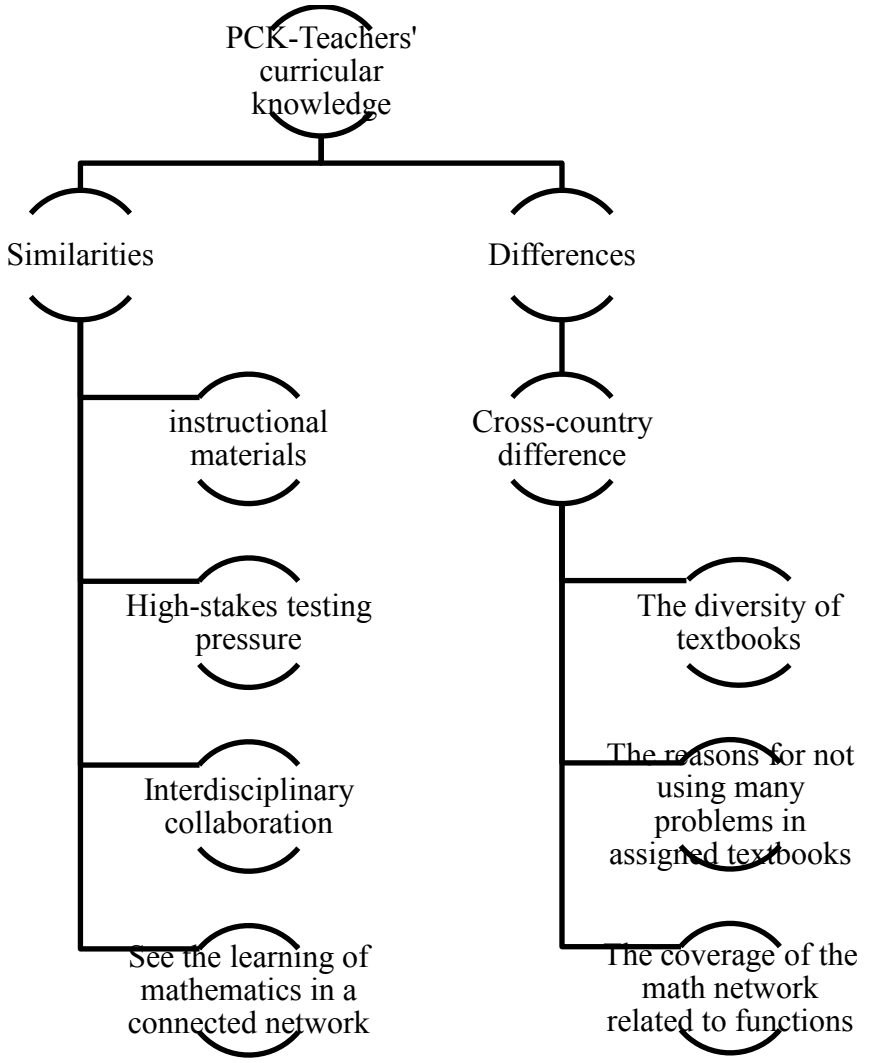


Figure 7 Chart of teachers' curricular knowledge of functions

CHAPTER 6

CONCLUSION AND IMPLICATIONS

Conclusion

The purpose of this study is to present the current state of the U.S. and Chinese middle school math teachers' pedagogical content knowledge for the topic of function. By dividing teachers of this study into four groups, i.e. the U.S. higher SES group, the U.S. lower SES group, the Chinese higher SES group, and the Chinese lower SES group, I am able to compare middle school math teachers' pedagogical content knowledge across countries (i.e. the U.S. and China) and socio-economic statuses (i.e. higher SES and lower SES). By examining teachers' response to questionnaire and teachers' introductory lesson plans of functions as well as interviews, I am able to obtain a better understanding of middle school math teachers' pedagogical content knowledge for functions as reflective of Shulman's (1986, 1987) identification of the important three components of PCK.

In the following sections, I summarize my findings on the current state of the U.S. and Chinese middle school math teachers' pedagogical content knowledge. I also provide implications at the end of this chapter.

Similarities in the current state of teachers' PCK of the study.

Teachers in this study, across countries and socio-economic statuses, share quite a few commonalities with respect to the three domains of PCK, i.e. instructional decisions, understanding of student mistakes and curricular knowledge.

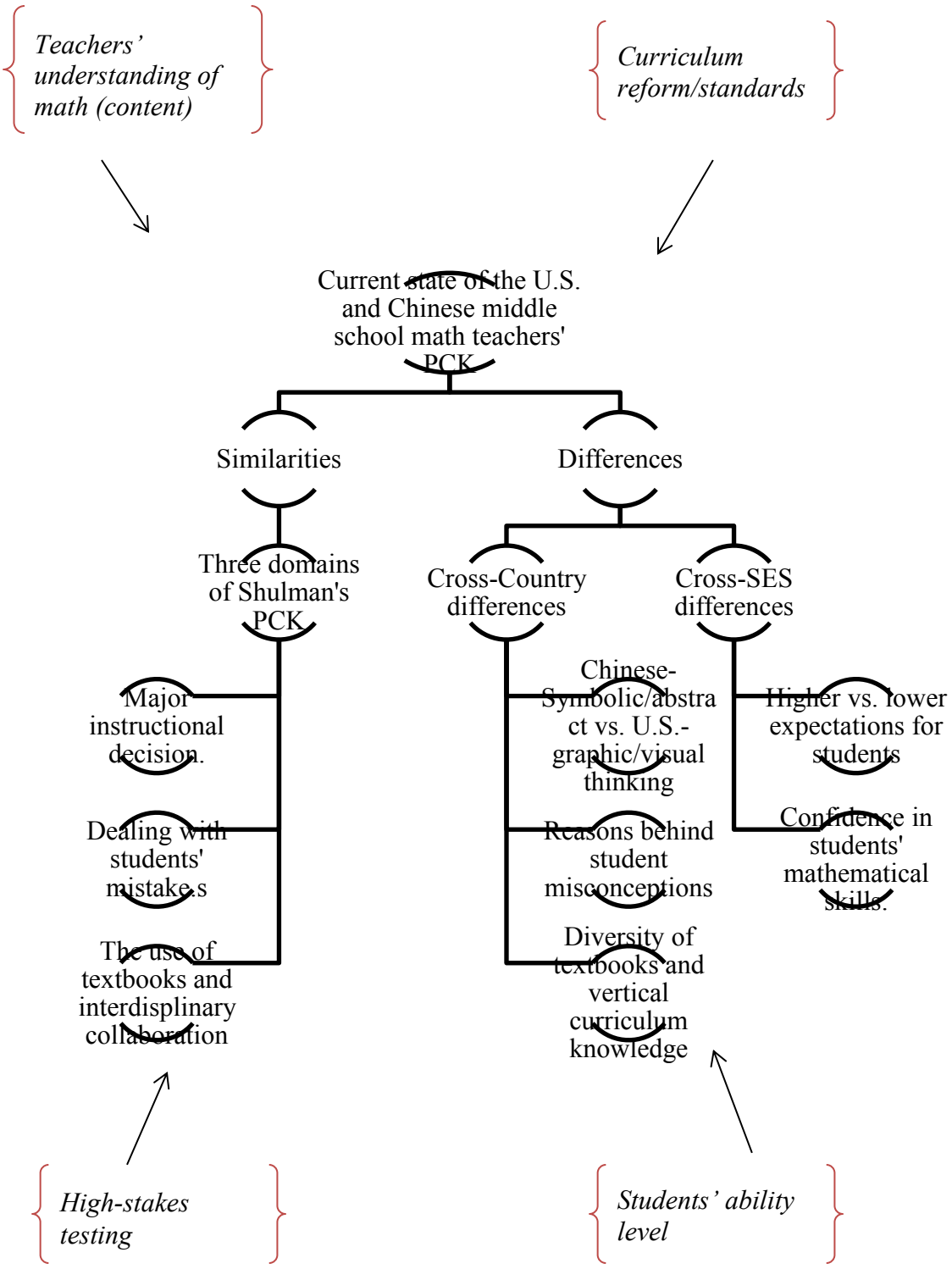


Figure 8 Current state of the U.S. and Chinese middle school math teachers' PCK

Teachers of this study are likely to make similar major instructional decisions on their introductory lesson of functions. First, they tend to expect students to understand the definition of function as a ready-made object in the introductory class, indicating that the structural approach of teaching is still common in the two cultures (see Sfard, 1991, 1992). Second, teachers of this study are all likely to construct mathematical tasks at the level of higher-cognitive-demand, focused on procedures with connections to concepts, understanding or meaning in class. This is consistent to Huang and Cai's (2010) argument that the U.S. and Chinese teachers are both likely to present their students with cognitively demanding tasks. It may reflect the positive influences of the curriculum reform efforts in the U.S. and China. Third, as influenced by the new standards in both countries, teachers of this study are inclined to encourage the use of multiple representations rather than one single representation of functions in class, a finding that is different from Friedlander and Tabach's (2001) argument that teachers do not often utilize multiple representations.

Teachers of this study also share some commonalities in dealing with students' mistakes. First, they tend to point out similar key concepts or ideas for correctly solving the math problems, which may reflect their shared understanding of the math problems themselves. Second, they tend to point out similar student misconceptions in solving the math problems. This indicates that teachers of this study has recognized the discrepancies between students' concept image and function definition (see similar findings in Tall & Vinner, 1981; Vinner & Dreyfus, 1989) and students' inclination to be over-literal in interpreting function graphs (see similar findings in Bell & Janvier, 1981; Monk, 1992). Third, teachers of this study also share some strategies in correcting students' mistakes.

They are equally likely to adopt straightforward (and easy) strategies in correcting students' mistakes. They are also likely to construct approaches which emphasize the conceptual understanding of functions when correcting students' mistakes.

In addition, teachers of this study share some commonalities in their curricular knowledge. They tend to use the same type of instructional materials (i.e. standards, textbooks, teachers-created materials and outside materials). Standards guide teachers' teaching practice in both the U.S. and China. Moreover, interdisciplinary collaboration is rarely seen in both the U.S and Chinese classrooms of this study, which indicates that math and other discipline (science in particular) are separated in their respective classrooms.

The similarities found in this study indicate that, despite cultural differences, teachers are likely to share a lot of commonalities regarding their current state of pedagogical content knowledge. These similarities, on the one hand, may reflect the dedication the U.S. and China make in improving math education. On the other hand, these similarities may also reflect the convergence in teaching practice brought by the educational reform efforts in both countries.

Difference in the current state of teachers' PCK of the study.

Despite similarities, teachers of this study show quite a few differences regarding teachers' PCK in teaching functions. These differences include cross-country differences and cross-SES differences.

What cross-country differences do the teachers of this study show regarding their pedagogical content knowledge? First, the U.S. and Chinese teachers of this study tend to place different values on the symbolic representations and graphic representations in both

making instructional decisions and correcting students' mistakes. Consistent with previous findings (see e.g., Cai & Wang, 2006; Mesiti & Clarke, 2010), while the U.S. teachers of this study are more likely to emphasize the graphic representations, the Chinese teachers are more likely to emphasize the symbolic representations in solving problems. This indicates that the U.S. teachers of this study may prefer the use of tangible, visual and intuitive math language in math teaching and learning, whereas the Chinese teachers may prefer the use of technical and abstract math language (see similar findings in An et al., 2004; Cai & Wang, 2006). This may reflect teachers' differential beliefs in the roles of different math languages in math education.

Second, teachers from the two countries compared in this study tend to show differential beliefs in the reasons underlying student misconceptions. While the U.S. teachers of this study are more likely to attribute student misconceptions of function graphs to their lack of experience of working with non-linear functions, the Chinese teachers of this study are more likely to attribute it to students' fixed-thinking after long-time linear-functions learning. It indicates that teachers of this study may hold differential beliefs in the natures of students' psychological development in math learning across the U.S. and China.

Third, the U.S. and Chinese teachers of this study diverge in the variety of textbooks and their vertical curriculum knowledge. On the one hand, the textbooks in the U.S. classrooms seem to be more diverse compared to that in the Chinese classrooms. This may reflect the different freedom degrees in choosing textbooks between the U.S. and Chinese middle school math teachers. On the other hand, how teachers of this study view the topic of function as connected to other math topics varies across countries. As

Ma (1999) argued in her research, the Chinese math teachers were inclined to present a more longitudinally cohesive "knowledge package" – a network related to specific content or problems. Consistent with her argument, I find the U.S. teachers of this study are inclined to view the topic of functions in a smaller system of patterns, relations and functions closely related to the topic, whereas the Chinese teachers are inclined to view the topic of functions in a more longitudinally cohesive and broader math knowledge network. This may reflect teachers' differential math knowledge packages between the two nations.

What cross-SES differences do the teachers of this study show regarding their pedagogical content knowledge? On the one hand, in this study, teachers from the higher SES schools are more likely to show higher expectations for their students compared to their counterparts from the lower SES schools. This is primarily reflected in teachers' differential instructional goals – while teachers from the higher SES school tend to expect their students to understand advanced concepts in introductory classes, teachers from the lower SES schools are more likely to expect their students to understand basic concepts.

On the other hand, teachers of this study tend to show differential confidence in their students' mathematical skills across socio-economic statuses. This is primarily reflected in that teachers from the higher SES schools are more likely to encourage their students to acquire higher skills in completing most cognitively demanding mathematical tasks, translating among representations and solving math problems, compared to their counterparts from the lower SES schools.

What might potentially influence teachers' PCK of the study?

Teachers' own subject-specific content knowledge determines their teaching practice (see Ball et al., 2008; Shulman, 1986, 1987). Consistent with this contention, I find in the present study that teachers' understanding of functions (and mathematics) plays a crucial role in teachers' pedagogical content knowledge of functions (and mathematics). In this study, teachers' shared understanding of functions may contribute to teachers' shared decisions on instruction and problem-solving. In the meanwhile, teachers' differential understanding of specific content may also contribute to their differences in the vertical curriculum knowledge related to that specific content.

Moreover, standards play an important part in teachers' pedagogical content knowledge. I find in the present study that standards (the *Arizona Common Core Standards* and the *New Standards* in Beijing) significantly influence teachers' instructional decisions and their understanding of instructional materials. This may reflect the values that teachers place on the standards, and also reflect the positive influences brought by the two nations' continuing efforts on the standards-based mathematics teaching reform. As standards-based instruction prevails in both the U.S. and Chinese classrooms, addressing the higher cognitive demand in the new standards has increasingly been important in classroom teaching (see also Huang & Cai, 2010; Wang & Lin, 2005).

High-stakes testing plays an important part in teachers' pedagogical content knowledge. Although the Chinese teachers of this study are more likely to be heavily influenced by testing compared to the U.S. teachers, high-stakes testing has been increasingly influencing teachers' classroom teaching. In this study, the

Arizona's Instrument to Measure Standards (AIMS) in the U.S. and the High School Entrance Exam in China not only influence teachers' selection of examples and problems used in classroom, but also influence to what degree they expect their students to understand a specific topic.

Lastly, as Shulman (1986, 1987) and other researchers (e.g., An et al., 2004; Carpenter et al., 1988) argued, understanding of students is key to teachers' pedagogical content knowledge. Congruent to these previous studies, I find that in this study, teachers' differential beliefs in students, especially students' ability levels significantly contribute to their differences in making instructional decisions and correcting students' mistakes across socio-economic statuses. The socio-economic status of the student population within a school is believed to be intertwined with the students' performance in that school: schools with affluent populations are more likely to be linked to higher student performances and schools with poor populations are more likely to be linked to lower student performances (see e.g., Feng & Lu, 2010; Hill et al., 2005; Hill, 2007; Hill & Lubienski, 2007; Zheng & Kahn, 2007). Consistent with this argument, teachers from the higher SES schools of this study tend to have stronger belief in their students' math ability compared to their counterparts from the lower SES schools of this study. Correspondingly, they are more likely to expect their students to understand more advanced concepts and acquire higher math skills, compared to those who are from the lower SES schools of this study.

Implications

Bridging the gaps in the literature.

The present study fills the gaps in the research literature from three aspects. First, this study provides thick descriptions (and comparisons) of experienced teachers' PCK in both the U.S. and China. Experienced teachers become the focus rather than being contrasted as "models" as they appeared in previous novice-expert comparisons. Second, this study presents a comprehensive analysis of math teachers' PCK, which was rarely found in prior research especially when it comes to China-U.S. comparisons. This study fills this gap through presenting the similarities and differences in teachers' PCK of instruction, of student understanding, and of curriculum. Third, this study casts light on middle school math teachers' PCK. Middle school math teachers' PCK has rarely been analyzed, especially through a comparative perspective. This study fills this gap through providing empirical information on middle school teachers in both the U.S. and China.

Policy implications.

The policy implications are twofold. First of all, the findings of the three components of pedagogical content knowledge in this study can be incorporated into the development of teacher education programs to improve math teachers' understanding of instruction, student thinking, and curriculum. In particular, the current state of experienced Chinese and the U.S. middle school math teachers' pedagogical content knowledge can inform the teachers' in-service training programs. For example, as discussed above, the teachers' knowledge of instruction, of student thinking and of curriculum are intertwined. From this perspective, teacher education program may help

teachers develop their pedagogical content knowledge in a systematic way rather than compartmentalize these components.

Second, continuing curriculum reform efforts are needed by both the policymakers and education practitioners. On the one hand, reform efforts involving textbooks may be put on the agenda because despite cultural differences, teachers in both countries show their concerns about the textbooks in the classrooms. On the other hand, high-stakes testing has stayed at the core of middle school education in China and has become increasingly important in school life in the U.S. The "conflicts" between the curriculum (textbooks, in particular) and those tests need to be addressed. Based on the findings of this study, continuing curriculum reform should be encouraged to make a more positive influence on teaching in each country.

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APPENDIX A

QUESTIONNAIRE

Background Information

1. Your gender:_____
2. What is your major in college? _____
3. How many years have you been teaching? _____
4. What teaching certificate do you have? _____
5. Currently, which school and what grade are you teaching? _____
6. Have you ever taught the concept of function? _____

   If yes, which books, materials, or resources have you used? _____

Problem Solving

Problem 1

If a student is asked to give an example of a graph of a function that passes through the points A and B She gives an example as shown in Figure 2. When asked if there is another answer for this question, she says "No".

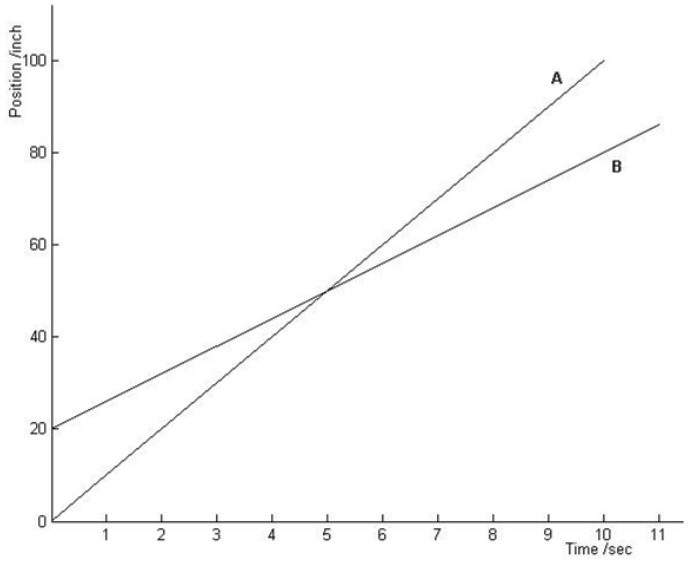


1. What are some of the important mathematical ideas that the student might use to answer this question correctly?

2. What do you think this student might be thinking? What underlying mathematical misconception(s) or misunderstanding(s) might lead student to the error?
3. How would you correct his/her misconception about determining functions with two points on coordinate plane?

Problem 2

A student is given the position vs. time graph as presented below. When asked to compare the speeds of the objects at time $t = 2$ sec., the student responds by saying that Object B is moving faster.



1. What are some of the important mathematical ideas that the student might use to answer this question correctly?
2. What do you think this student might be thinking? What underlying mathematical misconception(s) or misunderstanding(s) might lead student to the error?
3. How would you correct his/her misconception?

背景信息

1. 您的性别 _____

2. 您在大学的专业 _____

3. 您到目前为止教书的年数 _____

4. 您所持有的教师资格证为 _____

5. 您参加过任何的在职培训吗?_____

请列举_____

6. 目前，您在哪所中学_____ 教授几年级数学 _____

7. 您到目前为止教授过函数吗？

如果教过，请问您使用的教材是 _____

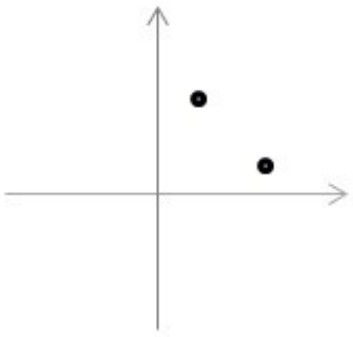
您还使用任何其他资料吗？请列举 _____

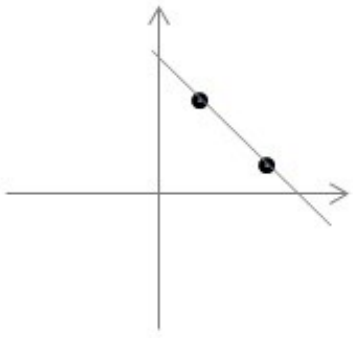
_____

开放式问题

问题一

一个学生被要求举一个经过下面左图两点的函数例子。她给出的例子如下面右图所示。当被继续追问经过这两点是否还可能有其他的函数时，她说“没有”。

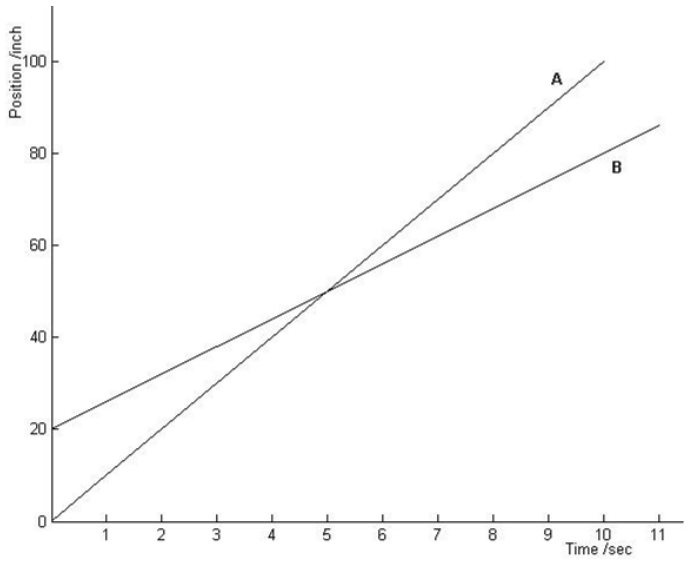




1. 要正确地回答这个问题，这个学生需要用到哪些重要的数学概念（知识）？
2. 你觉得这个学生回答这个问题的时候可能是怎么思考的？你觉得是什么理解上的误区导致这个学生犯这样的错误？
3. 你会怎么样来纠正她对坐标轴上两点确定函数问题的理解？

问题二

下图是两个物体 （A 和 B）的距离*时间函数图。一个学生被问到在第二秒（$t = 2$ sec）时哪个物体移动的速度快，她的回答是“物体 B”。



1. 要正确地回答这个问题，这个学生需要用到哪些重要的数学概念知识）？
2. 你觉得这个学生回答这个问题的时候可能是怎么思考的？你觉得是什么理解上的误区导致这个学生犯这样的错误？
3. 你会怎么样来纠正她对速度，时间，和距离问题的理解？

APPENDIX B

LESSON PLAN REQUEST

Imagine you are going to introduce the concept of function to your students next week. Please write a **complete** and **detailed** introductory lesson plan on the concept of function to show how you are going to teach this lesson. You can use your own template of lesson plan. When you write your lesson plan, please **at least** include the mathematical tasks you will construct for the lesson; the examples you will provide to teach the concept; and the assignment you will give to your students.

设想你将在下星期向你的学生介绍函数这一概念。请写一份**完整详细**的教案来展示你将如何介绍函数。你可以采用自己的教案模版。在你的教案中，请**至少**包括你将使用的数学任务；你将使用的函数实例；以及你将给学生布置的作业。

APPENDIX C

INTERVIEW PROTOCOL

Thank you for taking time to meet with me for this interview. Before we begin, I want to assure you that this interview will be used for research purposes only and your individual information will be kept confidential. This interview will take approximately an hour to an-hour-and-a-half. During the interview, you can stop anytime you want or you can choose not to respond to any questions if you feel uncomfortable. Do you have any questions about the interview process or the study? Are you okay with me recording this interview and taking some notes?

(Background questions)

For example,

1. How many years have you been teaching? How many years have you taught mathematics at middle school level? In the same school? No? In which school(s)?

2. Did you major in mathematics in college? No? What was your major college?

3. Which teacher preparation program did you go to? How long did it take? Tell me what you remember of your methods courses? Any methods courses related to math education? Have you taken any inservice training course or workshop? Any for math education?

(Turn to probe questions)

For example,

4. You indicated that you would have _____ (instructional goals) for your class. Could you please tell me why you set these goals?

5. You indicated that you would give your students _____ (representations of functions) in your lesson plan. Could you please tell me why you want to use _____ (representations)? Could you give them alternative representations? What

representations will you provide?

6. You mentioned you would use _____ (mathematical task) in your class. Could you please tell me why you want to use _____ (the mathematical task)? How do you think the task will help students understand the concept of function? Could you please tell me why you put this task at the beginning (in the middle, at the end) of the class? What activities will you organize to achieve the goal of this task? What representations will be used in this task? Why will you give these representations? Could you give other representations?

7. You indicated that you would give your student _____ (assignment or homework). Could you please tell me why you want to give them _____ (assignment)? How do you think this assignment will help students understand the concept of function?

8. What do you think that your students might be thinking when they make the mistakes? Why do they have this thinking or misconception?

9. How would you correct their mistakes or misconceptions? Why do you choose this approach?

10. What knowledge do you think is necessary for students to have in order to learn the concept of function? How do you find out whether your students have this knowledge or not? What difficulties do you think that your students might have regarding learning functions?

11. Could you please tell me the goal of learning functions in Arizona Mathematics Standards for middle school level? How do you understand this goal?

12. What instructional materials do you use for teaching functions? What textbooks do you use for teaching functions? What about other teachers you know? Do they use the same textbooks? Any other materials?

13. How do you view functions as related to other math topics? Do you know what your students will learn about functions in high school? Could you please give me some examples? What about before your formally introduced your students to the concept of function? What have they learned earlier to help them understand the concept of function? Could you please give me some examples?

14. Do you have any thoughts about why functions are formally taught at this grade? Do you collaborate with teachers from other disciplines? How do you collaborate? Could you please give me some examples?

15. Thanks again for meeting me for this interview. Lastly, do you have any other experiences of functions-related teaching that you would like share? Thanks!

感谢您抽出时间接受这次访谈。在我们开始之前，我想要向您说明，这次访谈将只用于研究，您的个人信息将完全保密。这次访谈大约需要占用您一到一个半小时的时间。在访谈开始之后，您可以随时选择终止访谈，或者选择不回答您不想回答的问题。您还有什么关于这个研究或者访谈过程的问题吗？您介意我录音和做一些笔记吗？

（背景信息问题）

比如：

1. 您教了多少年书了？

-教中学数学教了多少年了？

-一直在这个学校吗？ 如果不是，还在哪些学校待过？

2. 您大学是师范数学专业吗？

-如果是，那您还记得您都选了哪些教学方法课吗？您印象深的方法课程有吗？和数学教学相关的方法课程呢？

-如果不是，那您上岗前参加过任何专门的培训吗？有关数学教学方面的培训呢？

3. 您在职期间参加过任何的教师培训吗？什么样的培训？

（深度访谈问题）

比如：

4. 您给这节课制定的教学目标有哪些？为什么？

5. 您提到您将用到_____（数学任务）。

-您能告诉我您为什么要用这个_____（数学任务）吗？

-您觉得它是怎么帮助学生理解函数的概念的？

-您为什么要把它安排在这堂课的开始 （中间，结尾）呢？

-您具体会组织什么课堂活动来实现这个任务的目标呢？

-它会涉及到哪些具体的函数表达方式？您为什么会用这些函数表达方式？还有其他的吗？

6. 您提到您将会给学生布置_____（作业）。

- 您能告诉我您为什么要布置这些_____（作业）吗？

-您觉得这些作业怎么能帮助学生理解函数的概念？

-您觉得他们可能会用哪些方法来解答这个问题？

-您觉得学生们可能在这次作业中犯什么样的错误？

-您觉得他们为什么会犯这些错误？

-您会怎样纠正他们的错误？

7. 您提到您觉得学生可能会有这样一些误解，为什么呢？_____

8. 你怎么纠正它们的错误呢？

9. -您觉得学生们需要掌握什么知识基础来学习函数？

-您如何判断他们是否具备这个知识基础？

-您觉得他们在学习函数的时候可能存在什么错误的观念？

-您会如何帮助他们纠正这种错误的观念？

10. –您能告诉我新课程标准中对于初中函数的学习目标是什么吗？

-您如何理解这个目标？

11. –您现在用的是什么教材？

- 除了规定教材之外，您还有什么其他的材料来帮助教学吗？

12. –您觉得函数式如何和其他数学知识联系在一起的，函数在学生的数学体系中处在一个什么样的位置？

您知道您的学生高中阶段将会学习什么函数内容吗？

- 比如说？

- 在你正式向学生介绍函数概念之前，他们学习了什么和函数相关的概念？

13. – 您知道为什么函数这个概念要在初二的时候正式学习吗？

-您和其他学科的老师有合作吗？

其他的学科要用到函数吗？比如说物理，化学？您能举个例子吗？

14. 再次谢谢您抽出时间接受访谈。最后，您还有其他任何函数教学方面的经历想要和我分享吗？

谢谢！

APPENDIX D

CONTACT SUMMARY SHEET

Interviewee: Name Interviewee Number:

Date of Receiving Lesson Plan: Date of Receiving Questionnaire Response:

Date of Interviewing:

1. What were the main themes that stuck out in the lesson plan and questionnaire?

2. What particular interview questions should I pay attention to regarding this participant?

3. What discrepancies, if any, did I note in the participant's response in interview?

4. Is there anything else that stuck out as salient, interesting, or important in this contact?